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Abstract

"Numerical Methods of Parameter Identification for Problems Arising in Elasticity", by James Michael Crowley, Captain, USAF, Ph.D., Brown University, June, 1982, 130 pp.

Numerical methods for approximate identification or estimation of constant parameters in certain fourth-order partial differential equations (distributed parameter systems) from data are proposed based upon a reformulation of the problem as an abstract equation in a Hilbert space. Projections onto suitable subspaces of splines are used to obtain a semi-discrete approximation which is used to estimate the unknown parameters. Convergence of the approximations is proved using linear semigroup theory and the Trotter-Kato theorem. The proposed methods are applied to estimation of parameters in both the Euler-Bernoulli equation with structural and viscous damping and the Timoshenko equation for transverse vibration of a beam. Numerical results are presented.

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NUMERICAL METHODS OF PARAMETER IDENTIFICATION FOR PROBLEMS ARISING IN ELASTICITY

by

James Michael Crowley

A.B., College of the Holy Cross, 1971
M.S., Virginia Polytechnic and State University, 1973

Thesis

Submitted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy in the Division of Applied Mathematics at Brown University

June, 1982

This dissertation by James Michael Crowley is accepted in its present form by the Division of Applied Mathematics as satisfying the dissertation requirements for the degree of Doctor of Philosophy.

Date . 4/30/82 . . . .

Recommended to the Graduate School

Date April 30, 1982 . . . . George Majda . . . .

Date April 30, 1982 . . . .

Approved by the Graduate School

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Vita

James Michael Crowley was born on February 19th, 1949 in Baltimore, Maryland to George and Lorraine Crowley. He attended Loyola High School, graduating in 1971. He enrolled at the College of the Holy Cross in Worcester, Massachusetts, and received the Bachelor of Arts degree in mathematics in June, 1971. He then entered a Master of Science program in mathematics at Virginia Polytechnic Institute and State University, Blacksburg, Virginia, graduating in 1973. Subsequently, he entered the Air Force as a second lieutenant, employed as a numerical analyst at the Foreign Technology Division, WPAFB, Dayton, Ohio. He was married in 1973 to Anne Rafferty. From 1977 to 1979, he was an instructor in mathematics at the U.S. Air Force Academy. He began study in the Division of Applied Mathematics at Brown University in 1979. He is presently a captain in the U.S. Air Force, and a member of the Mathematical Association of America, the American Mathematical Society and the Society for Industrial and Applied Mathematicians.

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INTRODUCTION

In many problems of practical importance one would like to identify unknown parameters in mathematical models given certain observations of the underlying physical phenomenon being modeled. A general framework for approximating or estimating unknown parameters in partial differential equations, using modal (eigenfunction) approximations, was presented in [11]. The general theoretical framework developed in [11] was subsequently applied to spline-based techniques in [7] to two classes of second order initial-boundary value problems.

This thesis is devoted to developing numerical methods for estimating unknown constant parameters in certain fourth order constant coefficient partial differential equations. The approximation techniques follow the approach in [7] and convergence is proved using the theoretical convergence framework developed in [11] employing linear semigroup theory. The necessary theoretical framework is summarized in Chapter 1.

We treat identification problems for two specific equations in one dimension which model the transverse vibration of an elastic or viscoelastic beam, and develop numerical methods for estimating unknown parameters.

We examine estimation techniques for the Euler-Bernoulli equation in Chapter 2 and for the Timoshenko equation in Chapter 3. In both cases we develop methods for numerically estimating unknown parameters and prove convergence of the methods. Numerical results are provided to illustrate the theoretical convergence results.

The Euler-Bernoulli equation which we examine includes both structural (internal) and viscous damping, and various boundary conditions are used. We introduce two methods for estimating parameters, one based on quintic splines (Section 2) and one based on cubic spline approximations (Section 3). For the Timoshenko equation (Chapter 3) we examine one technique for estimation of parameters based upon cubic spline approximations.

In Chapter 4 we discuss the implementation of the approximation techniques into a computer code. Since all of the computer codes used here for the fourth order problems and in [7] for certain second order problems have the same general structure, th discussion of the implementation is made sufficiently general to describe all of the spline-based methods which we have developed.

The notation employed throughout this thesis is rather standard. For norms of elements in Banach spaces we use $|\cdot|$, whereas $||\cdot||$ is used for operator norms. A subscripted norm $|\cdot|_m$ denotes certain norms equivalent to the usual norms on the Sobolev spaces H^m over $[0,1]$, and specifically $|\cdot|_0$ denotes the H^0 (L^2) norm. Similarly, inner products on certain subspaces of the Sobolev spaces H^m will be denoted by $\langle\cdot,\cdot\rangle_m$. These will be defined in Section 2 of Chapter 1. As we shall be dealing with state spaces Z which are products of function spaces, the symbol $|\cdot|$ with no subscript (and similarly $\langle\cdot,\cdot\rangle$) will be reserved to denote the norm (or inner product) on the state space Z.

CHAPTER 1. FOUNDATIONS

Section 1. The Identification Problem and Its Approximation

We begin by defining the identification or estimation problem for a process governed by a partial differential equation and proceed to its abstract formulation. A general framework for approximating solutions is introduced along with tools necessary to establish convergence. The techniques for approximating solutions to the identification problem follow those introduced in [11], where modal (eigenfunction) state approximations were applied to a class of hyperbolic and parabolic equations, and also used in [7], where spline-based state approximations were applied to the same class of problems.

This section outlines the basic approach and theory which will be applied in later chapters to two specific equations, namely the Euler-Bernoulli and Timoshenko equations for the transverse vibration of a beam, and to specific spline-based approximations for the identification problem.

We first define the identification problem for an initial-boundary value problem. Suppose we have a physical process modeled by an initial-boundary value problem with unknown parameters $q = (q_1,\ldots,q_p) \in R^p$; the parameters q_i may be unknown constant coefficients in the partial differential equation or parameters appearing in functions in the initial conditions or non-homogeneous term. Suppose also that we are given a set of output measurements from the physical process which is modeled by the initial-boundary value problem. In a sense which will be made more precise in what follows, the identification (or estimation) problem consists of finding the vector of parameters $q \in R^p$ such that the solution of the initial-boundary value problem "best fits" the output measurements of the physical process.

We will be interested in two particular initial-boundary value problems, namely the Euler-Bernoulli and Timoshenko equations for the transverse vibration of a uniform beam. The Euler-Bernoulli equation, including structural and viscous damping, has the form

$$\text{(1.1)} \qquad \begin{aligned} y_{tt} &= -q_1 D^4 y - q_2 D^4 y_t - q_3 y + f(t,x;q), \quad t > 0,\ x \in [0,1] \\ y(0,x) &= y_0(x;q) \\ y_t(0,x) &= y_1(x;q), \end{aligned}$$

with appropriate homogeneous boundary conditions at $x = 0$ and $x = 1$, where $D^j \equiv \partial^j/\partial x^j$, $y(t,x;q)$ is the transverse displacement, and $f(t,x;q)$ is the applied load. Here, q_1, q_2, and q_3 are unknown constant coefficients, and $q_4,\ldots,q_p$ are parameters appearing in the nonhomogeneous (load) term and initial conditions.

The Timoshenko equations can be put in the form

$$\text{(1.2)} \qquad \left.\begin{aligned} y_{tt} &= q_1 D^2 y - q_1 D\psi + f(t,x;q) \\ \psi_{tt} &= q_3 D^2 \psi + q_2(Dy - \psi) \end{aligned}\right\} \quad t > 0,\ x \in [0,1]$$

$$\begin{aligned} y(0,x) &= y_0(x;q) \\ y_t(0,x) &= y_1(x;q) \\ \psi(0,x) &= \psi_0(x;q) \\ \psi_t(0,x) &= \psi_1(x;q) \end{aligned}$$

where again we associate appropriate homogeneous boundary conditions with (1.2) at $x = 0$ and $x = 1$. Here $y(t,x;q)$ is transverse displacent and $\psi(t,x;q)$ is the angle of rotation of a cross-section of the beam.

More will be said about these equations in the sequel. The above is sufficient at present to set the framework for the identification problem.

We shall be interested in the identification problem associated with (1.1) or (1.2). Given a set of observations $\hat{\eta} = \{\hat{\eta}_i\}_{i=1}^{r}$, where $\hat{\eta}_i = (\hat{y}(t_i,x_1),\ldots,\hat{y}(t_i,x_\ell))^T$ and $\hat{y}(t_i,x_j)$ is the observed displacement at t_i, x_j, of a process which we assume to be modeled by (1.1) or (1.2), find the vector of parameters $\bar{q} = (\bar{q}_1,\ldots,\bar{q}_p)$ which minimizes $J(q;y,\hat{\eta})$, where J is some functional measuring the fit of (1.1) or (1.2) to the data $\hat{\eta}_i$. Specifically, we shall use a pointwise fit-to-data criterion of the form

$$(1.3) \qquad J(q,y,\hat{\eta}) = \sum_{n=1}^{r} |\xi_i - \hat{\eta}_i|^2$$

where $\xi_i = (y(t_i,x_1),\ldots,y(t_i,x_\ell))^T$, and $(t,x) \longmapsto y(t,x)$ (respectively, $(t,x) \longmapsto (y(t,x),\psi(t,x))$ is the solution of (1.1) (respectively, of (1.2)).

To ensure that the initial-boundary value problems are well-posed, *we shall assume hereafter*

1) Q is a compact set in R^p, and

(HQ) 2) there exists $q_1^L > 0$ such that $q_1 \geq q_1^L$ for all $q \in Q$.

For the Timoshenko equations, we also require, along with the above

2') there exists $q_3^L > 0$ such that $q_3 \geq q_3^L$ for all $q \in Q$.

Remark. Note that we have formulated the initial-boundary value problem for homogeneous boundary conditions. If nonhomogeneous time dependent boundary conditions occur, of the form $(D^j y)|_{x=0} = h(t,q)$ they will be transformed to homogeneous boundary conditions by a transformation of the form $v(t,x) = y(t,x) - g(x)h(t,q)$, (see [29], where a general method for transforming (1.1) with time-dependent boundary conditions to (1.1) with homogeneous boundary conditions, in the case $q_2 = 0$, $q_3 = 0$, is discussed). When such a transformation is performed and the output measurements in the fit-to-data criterion correspond, as above, to y, it is appropriate to modify J in (1.3) to

$$(1.3') \qquad J'(q,v,\hat{\eta}) = \sum_{i=1}^{r} |\xi_i' - \hat{\eta}_i|^2,$$

where

$$\xi_i' = (v(t_i,x_1) + g(x_1)h(t_i,q),\ldots,v(t_i,x_\ell) + g(x_\ell)h(t_i;q))^T.$$

Definition 1.1. The identification problem (ID) for (1.1) (or (1.2)) is defined as the following: given (1.1) with unknown parameters $q = (q_1,\ldots,q_p) \in R^p$ and a family of solutions $(t,x) \longmapsto y(t,x;q)$ $((y(t,x;q),\psi(t,x;q))^T$ for (1.2)), and a set of output measurements $\{\hat{\eta}(t_i)\}_{i=1}^r$, find $\bar{q} \in Q$, where Q is some parameter set in R^p satisfying (HQ), such that

$$J(\bar{q},y(\cdot,\cdot;\bar{q}),\hat{\eta}) \leq J(q,y(\cdot,\cdot;q),\hat{\eta}) \quad \text{for all} \quad q \in Q,$$

where J is the cost functional (1.3).

Having formulated the identification problem for two model initial-boundary problems, we proceed to an abstract formulation of the problem. First we write the initial-boundary value problem (1.1) or (1.2) as an abstract equation on an appropriate Hilbert space Z, in the form

$$(1.4) \qquad \begin{aligned} \dot{z}(t) &= \mathscr{A}(q)z(t) + F(q,t) \quad \text{for} \quad t > 0 \\ z(0) &= z_0 \end{aligned}$$

where $z(t) = (z_1(t),\ldots,z_n(t)) \in Z$ for every $t \in [0,T]$, and $z_1(t)$ corresponds to $y(t,\cdot)$ in (1.1) or (1.2).

Specific formulations of (1.1) and (1.2) as an abstract equation will be examined in Chapters 2 and 3. In each case, the abstract equation will be formulated in such a way that $\mathscr{A}(q)$ generates a C_0 semigroup $T(t;q)$ on Z; i.e., $t \longmapsto T(t,q)z_0$ is the solution to $\dot{z}(t) = \mathscr{A}(q)z(t)$, $z(0) = z_0$.

The notion of dissipativeness will play a central role in proving that $\mathscr{A}(q)$ is generator of a C_0 semigroup in the specific examples to be considered. A densely defined operator $\mathscr{A}$ is called <u>dissipative</u> if $\langle \mathscr{A}z,z\rangle \leq 0$ for all $z \in \mathrm{Dom}(\mathscr{A})$, and <u>maximal dissipative</u> if its only dissipative extension is itself. While there are many simple conditions which guarantee that a dissipative operator generates a C_0 semigroup, we shall use one primarily: a maximal dissipative operator $\mathscr{A}$ generates a C_0 semigroup $\{T(t)\}$ of contraction on Z; i.e., $\|T\| \leq 1$ (cf: [25, p. 88]).

We will be interested in mild solutions of (1.4): $t \longmapsto z(t;q)$ is called a mild solution on $[0,T]$ if it satisfies

$$(1.5) \qquad z(t;q) = T(t;q)z_0 + \int_0^t T(t-s;q)F(q,s)ds$$

for every $t \in [0,T]$. We will also place the following conditions on F which guarantee uniqueness and existence of mild solutions $z(\cdot,q) \in C(0,T;Z)$ to (1.4) (see [11, p. 13]):

(HF)
1) The map $t \longmapsto F(q,t)$ is measurable,
2) the map $q \longmapsto F(q,t)$ is continuous, and
3) there exists $k(t) \in L^2(0,T)$ such that $|F(q,t)| \leq k(t)$.

We note in passing that when strong (classical) solutions to (1.4) exist they will be mild solutions. In fact, if $z_0 \in \mathrm{Dom}(\mathscr{A}(q))$ and $t \longmapsto F(q,t)$ is strongly continuously differentiable in $(0,T)$ with derivative continuous in $[0,T]$, then, [, p. 203]

i) $z(t)$ is absolutely continuous in $(0,T)$, where $t \longmapsto z(t)$ satisfies (1.4)

ii) $z(t) \in \mathrm{Dom}(\mathscr{A})$ for $t > 0$

iii) $|z(t) - z_0| \to 0$ as $t \to 0$

iv) $z(t)$ satisfies (1.5); i.e., $t \longmapsto z(t)$ is a mild solution.

Also, the relation between weak solutions of (1.4) and mild solutions of (1.4) is given by the following theorem [3, p. 204]. If we relax the assumptions above and only require that $F(q,\cdot) \in L^2(0,T;z)$, then there exists a unique weak solution $t \longmapsto z(t;q)$ of (1.4) for $0 \leq t \leq T$, where

$$\langle z(t),\zeta\rangle \text{ is absolutely continuous for every } \zeta \in \mathrm{Dom}(\mathscr{A}^*),$$

and

$$\frac{d}{dt}\langle z(t),\zeta\rangle = \langle z(t),\mathscr{A}^*\zeta\rangle + \langle F(q,t),\zeta\rangle , \quad 0 < t < T.$$

Furthermore, $z(t)$ satisfies (1.5); i.e., $t \longmapsto z(t)$ is also a mild solution.

To define point evaluations, we note that Z will always be taken as a function space of R^n-valued functions defined on $[0,1]$. Thus for every $q \in Q$, $t \in [0,T]$, we may associate the function $u(\cdot,x;\cdot): [0,1] \to R^n$ satisfying $u(t,\cdot;q) = z(t;q)$. Then, with $u(t,\cdot;q) = (u_1(t,\cdot;q),\ldots,u_n(t,\cdot;q))$, we define the map $\pi_0 z(t;q) \equiv u_1(t,\cdot;q)$. In subsequent applications, we will always take $Z = Z_1 \times \ldots \times Z_n$ where Z_1 consists of functions sufficiently smooth (in fact, subspaces of the Sobolev spaces H^m, where $m \geq 1$) that point evaluations of u_1 will make sense. Now, as we have set up the abstract equation (1.4) so that $z_1(t;q)$ corresponds to $y(t,\cdot;q)$, we may define the cost functional $\tilde{J}(q,z(\cdot,q),\hat{\eta})$ by

$$(1.6) \qquad \tilde{J}(q,z(\cdot;q),\hat{\eta}) = J(q,\pi_0 z(\cdot,q),\hat{\eta}),$$

where J is defined in (1.3).

This leads us to define in place of (ID), the abstract identification problem.

<u>Definition 1.2</u>. The <u>abstract identification problem (IDA)</u> consists of the following: find $\bar{q} \in Q \subset R^p$ such that

$$\tilde{J}(\bar{q},z(\cdot,\bar{q}),\hat{\eta}) \leq \tilde{J}(q,z(\cdot,q),\hat{\eta}) \text{ for all } q \in Q,$$

subject to $z(\cdot;q)$ satisfying (1.5).

<u>Remark</u>. This is, of course, a reformulation of the original identification problem in terms of the abstract equation, employing mild solutions. The two coincide where classical solutions of (1.1) or (1.2) exist. It is possible (see [11, pp. 7-8]) to formulate the cost functional $\tilde{J}$ in a more general way so as to permit identification when data $\hat{\eta}$ consists of measurements of

quantities other than transverse displacements. Our goal here is to demonstrate how a numerical approximation may be constructed, based on the form of (1.4) and the choice of Z, and hence we have restricted our attention to the above problem.

Since we cannot solve (IDA) directly in general, we describe a general procedure for approximating solutions to (IDA) following the approach taken in [7] and [11]. We take a sequence of finite dimensional subspaces $Z^N \subset Z$ and define $P^N: Z \to Z^N$ to be the orthogonal projection satisfying

(1.7) $\quad |P^N z - z| \le |\zeta - z|$ for all $\zeta \in Z^N$, or equivalently,

(1.8) $\quad \langle P^N z - z, \zeta \rangle = 0$ for all $\zeta \in Z^N$.

The subspaces Z^N will be chosen so that $Z^N \to Z$ in the sense that $P^N \to I$ *strongly on* Z. We then replace the abstract equation (1.4) on Z by a sequence of approximating equations on Z^N:

(1.9) $$\dot{z}^N(t) = \mathscr{A}^N(q) z^N(t) + F^N(q,t) \quad \text{for} \quad t > 0$$
$$z^N(0) = z_0^N(q).$$

The approximations we use are

$$\mathscr{A}^N(q) = P^N \mathscr{A}(q) P^N$$
$$F^N(q,t) = P^N F(q,t)$$
$$z_0^N = P^N z_0(q).$$

Note that this choice of $\mathscr{A}^N(q)$ requires that $Z^N \subset \mathrm{Dom}(\mathscr{A}(q))$. This requirement not only imposes limitations on the smoothness of the elements in Z^N but also dictates that the boundary conditions, which appear in $\mathrm{Dom}(\mathscr{A}(q))$, be satisfied by every $z^N(t) \in Z^N$.

Remark. The norms on the spaces Z used in subsequent applications of this theory may be parameter dependent, in which case the projections $P^N(q)$ may depend on the parameters. Although we do not emphasize this parameter dependence until we encounter specific cases, this should be kept in mind.

Assuming that $\mathscr{A}(q)$ is the generator of a C_0 semigroup, $\mathscr{A}(q)$ is closed, and by the closed graph theorem, $\mathscr{A}^N(q)$ is a bounded operator (see [3, p. 80]. $\mathscr{A}$ closed and B bounded implies $\mathscr{A}B$ is bounded, hence $P^N\mathscr{A}(q)P^N$ is a bounded operator). Thus $\mathscr{A}^N(q)$ generates a semigroup $\{T^N(t;q)\}$ given by $e^{\mathscr{A}^N(q)t}$. Moreover, $t \longmapsto z^N(t;q)$ satisfies (1.9) on $[0,T]$ if and only if it satisfies

$$(1.10) \qquad z^N(t;q) = T^N(t;q)P^N z_0(q) + \int_0^t T^N(t-s;q)P^N F(q,s)ds.$$

Then standard Picard iteration arguments imply that solutions $z^N(\cdot;q)$ of (1.10) exist.

Moreover, when $\mathscr{A}(q)$ is maximal dissipative (and hence generates a C_0 semigroup of contractions), then $\mathscr{A}^N(q)$ will also be maximal dissipative, and so $\|T^N(t;q)\| \leq 1$. This follows from the following:

$$\begin{aligned} \langle\mathscr{A}^N(q)z,z\rangle &= \langle P^N\mathscr{A}(q)P^N z,z\rangle \\ &= \langle\mathscr{A}(q)P^N z,P^N z\rangle \\ &\leq 0 \end{aligned}$$

and

$$\begin{aligned}<(\mathscr{A}^N(q))^*z,z> &= <(P^N\mathscr{A}(q)P^N)^*z,z> \\ &= <P^N(\mathscr{A}(q))^*P^Nz,z> \\ &= <(\mathscr{A}(q))^*P^Nz,P^Nz> \\ &\le 0,\end{aligned}$$

where we have employed the self-adjointness of the projection operator P^N and the characterization that a dissipative operator $\mathscr{A}$ is maximal dissipative if and only if $<\mathscr{A}z,z> \le 0$, $<\mathscr{A}^*z,z> \le 0$ and $\mathscr{A}$ is closed [25, p. 85].

We can then formulate an approximate identification problem.

Definition 1.3. The approximate identification problem (IDN) consists of the following: find $\bar{q}^N \in Q \subset R^p$ such that

$$\tilde{J}(\bar{q}^N,z^N(\cdot;\bar{q}^N),\hat{n}) \le \tilde{J}(q,z^N(\cdot;q),\hat{n}) \quad \text{for all} \quad q \in Q$$

subject to $z^N(\cdot;q)$ satisfying (1.10) on $[0,T]$.

One of the goals will be to prove convergence of the solutions $\bar{q}^N$ of the approximate problem (IDN) to a solution $\bar{q}$ of the identification problem (IDA). In [11, pp. 15-17], it was proved that the map $q \longmapsto z^N(t;q)$ is continuous for $z^N(\cdot;q)$ satisfying (1.10). Since $\tilde{J}$ consists of point evaluations on the first component of $z^N(\cdot;q)$, it is easily argued that the map $v \longmapsto \tilde{J}(\cdot,v,\cdot)$ is continuous on $C([0,T];z)$ (recall again that we will take $Z = Z_1 \times \ldots \times Z_n$ such that Z_1 is a subset of H^1 or H^2). Thus we find that the map $q \longmapsto \tilde{J}(q,z^N(\cdot;q),\hat{n})$ is continuous, and so for each N there exists a solution $\bar{q}^N$ to the approximate identification problem (IDN). By the compactness of Q, there exists a convergent subsequence (again denoted by $\bar{q}^N$) such that $\bar{q}^N$ converges to some $\bar{q}$ in Q.

We show that this $\tilde{q}$ is a solution to the identification problem (IDA). This will be accomplished if we can show

$$(1.11) \quad \lim_{N\to\infty} |q^N - q^*| = 0 \quad \text{implies} \quad \lim_{N\to\infty} |z^N(t;q^N) - z^N(t;q^*)| = 0$$

for every $t \in [0,T]$. To see this, observe that $\tilde{J}(\bar{q}^N, z^N(\cdot;\bar{q}^N),\hat{\eta}) \le \tilde{J}(q, z^N(\cdot;q),\hat{\eta})$ for every $q \in Q$, since $\bar{q}^N$ is a solution of (IDN). Taking the limit as $N \to \infty$, and applying (1.11), we obtain

$$J(\bar{q}, z(\cdot;\bar{q}),\hat{\eta}) \le J(q, z(\cdot;q),\hat{\eta})$$

for every $q \in Q$. Thus $\bar{q}$ will be a solution of the identification problem (IDA) if (1.11) can be shown to hold. Since $t \longmapsto z(t;q)$ satisfies

$$(1.4) \quad z(t;q) = T(t;q)z_0(q) + \int_0^t T(t-s;q)F(q,s)ds, \quad t \in [0,T],$$

and $t \longmapsto z^N(t;q)$ satisfies

$$(1.10) \quad z^N(t;q) = T^N(t;q)P^N z_0(q) + \int_0^t T^N(t-s;q)P^N F(q,s)ds, \quad t \in [0,T],$$

an application of the Lebesgue bounded convergence theorem (see [11, p. 20]) yields a convenient criterion for showing when (1.11) holds.

<u>Proposition 1.4</u>. Let $t \longmapsto z(t;q)$ and $t \longmapsto z^N(t;q)$ be solutions of (1.4) and (1.10) respectively, and assume (HF) holds. Then

$$\lim_{N\to\infty} |q^N - q^*| = 0 \quad \text{implies} \quad \lim_{N\to\infty} |z^N(t;q^N) - z(t;q^*)| = 0$$

for every $t \in [0,T]$ if

i) $\|T^N(t;q^N)\| \leq Me^{\omega t}$ with M,ω independent of N, q.

ii) $P^N \to I$ strongly in Z.

iii) $T^N(t;q) \to T(t;q^*)$ strongly in Z and uniformly in $t \in [0,T]$ when $\{q^N\}$ is any convergent sequence with $q^N \to q^*$.

This proposition, proved in [11, p. 20] for the more general case when F can be mildly nonlinear, is the fundamental tool in proving convergence of solutions of the approximate identification problem to solutions of the abstract identification problem. Verification of (i) has already been treated in the case where $\mathscr{A}(q)$ is the generator of a C_0 semigroup of contractions. In that case, we have already shown that $\|T^N(t;q)\| \leq 1$.

Part (ii) of Proposition 1.4 will be verified for the specific cases where Z^N is the linear span of certain cubic or quintic splines satisfying prescribed boundary conditions. This will be done in Section 3.

Finally, part (iii) of Proposition 1.1 can be established using the Trotter-Kato theorem, which can be viewed as a functional analytic version of the Lax Equivalence Theorem (stability plus consistency implies convergence). The version we use is due to Kurtz [28]:

<u>Proposition 1.5</u>. [28]. Let $(\mathscr{B},|\cdot|)$ and $(\mathscr{B}^N,|\cdot|_N)$, $N = 1, 2,\ldots,$ be Banach spaces and let $\pi^N\colon \mathscr{B}\to\mathscr{B}^N$ be bounded linear operators. Assume further that $\mathscr{T}(t)$ and $\mathscr{S}^N(t)$ are linear C_0 - semigroups on $\mathscr{B}$ and $\mathscr{B}^N$ with infinitesimal generators $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{A}}^N$ respectively. If

i) $\lim_{N\to\infty} |\pi^N z|_N = |z|$ for all $z \in \mathscr{B}$,

ii) there exist constants $\tilde{M},\tilde{\omega}$ independent of N such that $\|\mathscr{T}^N(t)\|_N \leq \tilde{M}e^{\tilde{\omega}t}$, for $t \geq 0$,

iii) there exists a set $\mathscr{D} \subset \mathscr{B}$ such that $\mathscr{D} \subset \mathrm{dom}(\tilde{\mathscr{A}})$, $\bar{\mathscr{D}} = \mathscr{B}$ and $(\lambda_0 - \tilde{\mathscr{A}})\mathscr{D} = \mathscr{B}$ for some $\lambda_0 > 0$ for which for all $z \in \mathscr{D}$ we have $\lim_{N\to\infty} |\tilde{\mathscr{A}}^N \pi^N z - \pi^N \tilde{\mathscr{A}} z|_N = 0$,

then $\lim_{N\to\infty} |\mathscr{T}^N(t)\pi^N z - \pi^N \mathscr{T}(t) z|_N = 0$ for all $z \in \mathscr{B}$, uniformly in t on compact subsets of $[0,\infty)$.

Section 2. Preliminary Definitions of Spaces and Norms

We now introduce some of the spaces and norms to be used in later chapters and collect some facts about them. Denote by H^m the usual Sobolev spaces over $[0,1]$ in one dimension. These are

Definition 1.6.

$$H^m = \{\phi : D^{m-1}\phi \text{ is absolutely continuous on } [0,1] \text{ and } D^m\phi \in L^2(0,1)\}.$$

The usual norm on H^m is $|\phi|^2_{H^m} = \sum_{j=0}^{m} |D^j\phi|^2_{L^2}$. Thus, $H^0 = L^2$. The spaces H^m with this norm are Hilbert spaces [47, p. 55; 1, pp. 44-47]. Moreover, the graph norm defined by

$$|\phi|^2_G = |\phi|^2_{L^2} + |D^m\phi|^2_{L^2}$$

is an equivalent norm to $|\cdot|_{H^m}$ [1, p. 79]. Denote the norm in H^0 by $|\cdot|_0$.

Since we will require functions satisfying certain prescribed boundary conditions, we will be concerned with certain subspaces of H^m. We will also find it convenient to weight the norm in those spaces in order to obtain dissipative estimates. In particular, define for a fixed $\alpha \in R$, $\alpha > 0$

$$H_0^m(\alpha) = \{\phi \in H^m\colon \phi(0) = \phi(1) = 0,\ (D\phi)(0) = (D\phi)(1) = 0, \ldots, (D^{m-1}\phi)(0) = (D^{m-1}\phi)(1) = 0\}.$$

with inner product

$$\langle \phi, \psi \rangle_{m,\alpha} \equiv \langle \alpha D^m \phi, D^m \psi \rangle_0,$$

where $\langle \cdot, \cdot \rangle_0$ is the H^0 (L^2) inner product. Denote the norm induced by this inner product as $|\cdot|_{m,\alpha}$. Then $H_0^m(\alpha)$ is a Hilbert space and the norm $|\cdot|_{m,\alpha}$ is equivalent to the usual norm $|\cdot|_{H^m}$ on H_0^m (cf: [1, p. 158]).

We will also need certain other subspaces of H^2.

<u>Definition 1.7</u>. Given $\alpha > 0$, define

$$H_1^2(\alpha) \equiv \{\phi \in H^2\colon \phi(0) = \phi(1) = 0\}$$

$$H_2^2(\alpha) \equiv \{\phi \in H^2\colon \phi(0) = \phi(1) = \phi'(0) = \phi'(1) = 0\}$$

$$H_3^2(\alpha) \equiv \{\phi \in H^2\colon \phi(0) = \phi'(0) = 0\}$$

where $H_k^2(\alpha)$ is equipped with the inner product

$$\langle \phi, \psi \rangle_{2,\alpha} = \langle \alpha D^2 \phi, D^2 \psi \rangle_0 .$$

Note that $H_2^2(\alpha) = H_0^2(\alpha)$ as defined above. When $\alpha = 1$, we simplify the notation by writing $H_k^2 \equiv H_k^2(1)$.

<u>Theorem 1.8</u>. The spaces H_k^2 are Hilbert spaces and the norm

$|\cdot|_{2,1}$ is equivalent to the usual norm $|\cdot|_{H^2}$ on H_k^2 for $k = 1,2,3$.

Proof: The case $k = 2$ is well known (see above). We consider the cases $k = 1$ and $k = 3$. First we observe that $|\cdot|_{2,1}$ is a norm on H_k^2:

Since $|\cdot|_{2,1}$ is a semi-norm on H^2 [1, p. 73], we only need to show $|u|_2 = 0$ implies $u = 0$ for $u \in H_k^2$, $k = 1,3$. But $|u|_2 = 0$ implies $\int_0^1 (D^2u)^2 = 0$, which implies $D^2u = 0$ a.e., or $u = ax + b$ a.e.; furthermore, u, Du are absolutely continuous, $u(0) = u(1) = 0$ (if $k = 1$) or $u(0) = u'(0) = 0$ (if $k = 3$), which implies $a = b = 0$.

1) $k = 1$. We first prove H_1^2 is a closed subspace of H^2. Suppose $u_n \in H_1^2$ is a Cauchy sequence in the $|\cdot|_2$ norm; then $u_n \to u_0 \in H_1^2$. But $u_n \in H_1^2$ implies that $u_n(0) = u_n(1) = 0$ for every n. Thus, $u_n \to u$ in H^2 implies that $u_n(0) \to u(0)$, and $u_n(1) \to u(1)$, and so $u(0) = u(1) = 0$.

Next we prove $|\cdot|_2$ is equivalent to the $|\cdot|_G$ norm (and hence to the $|\cdot|_{H^2}$ norm.

Let $u \in H_1^2$. Then there exists a $v \in H^0$ such that $D^2u = v$ and for every $v \in H^0$ there exists $u \in H_1^2$ such that $D^2u = v$, namely $u(x) = \int_0^x \int_0^{s_1} v(s)dsds_1 - x \int_0^1 \int_0^{s_1} v(s)dsds_1$. Then, by the Rayleigh-Ritz inequality [40, p. 5],

$$\begin{aligned}|u|_0 &\le \frac{1}{\pi} |Du|_0 \\ &= \frac{1}{\pi}\left|D\left(\int_0^x \int_0^{s_1} v(s)dsds_1 - x \int_0^1 \int_0^{s_1} v(s)dsds_1\right)\right|_0 \\ &= \frac{1}{\pi}\left|\int_0^x v(s)ds - \int_0^1 \int_0^{s_1} v(s)dsds_1\right|_0\end{aligned}$$

$$\leq \frac{1}{\pi}\left\{\left|\int_0^x v(s)ds\right|_0 + \left|\int_0^1 \int_0^{s_1} v(s)dsds_1\right|_0\right\}.$$

Since $v \in H^0$, Schwartz's inequality implies $\int_0^1 |v(s)|ds \leq |v|_0$. Define $g(x) \equiv \int_0^x |v(s)|ds \leq \int_0^1 |v(s)|ds \leq |v|_0$ for all $x \in [0,1]$. Let $c \equiv |v|_0$. Then $\int_0^1 g(s)ds \leq \int_0^1 c = c$. So, continuing the chain of inequalities, we obtain

$$|u|_0 \leq \frac{1}{\pi}\left\{\left|\int_0^x v(s)ds\right|_0 + \left|\int_0^1 \int_0^{s_1} v(s)dsds_1\right|_0\right\}$$

$$\leq \frac{1}{\pi}\left\{\left|\int_0^x |v(s)|ds\right|_0 + \left|\int_0^1 \int_0^{s_1} |v(s)|dsds_1\right|_0\right\}$$

$$\leq \frac{1}{\pi}\{|c|_0 + |c|_0\} = \frac{2}{\pi}c = \frac{2}{\pi}|v|_0.$$

But $v = D^2u$ a.e., so

$$|u|_0 \leq \frac{2}{\pi}|v|_0 = \frac{2}{\pi}|D^2u|_0 = \frac{2}{\pi}|u|_2,$$

and

$$|D^2u|_0^2 \leq |D^2u|_0^2 + |u|_0^2 \leq |D^2u|_0^2 + \frac{4}{\pi^2}|D^2u|_0^2$$
$$= (1 + \frac{4}{\pi^2})|D^2u|_0^2,$$

or

$$|u|_2^2 \leq |u|_G^2 \leq (1 + \frac{4}{\pi^2})|u|_2^2.$$

Now, since the graph norm $|\cdot|_G$ is equivalent to the usual H^2 norm $|\cdot|_{H^2}^2 = \sum_{j=0}^{2} |D^ju|_0^2$ on H^2, we have equivalence of $|\cdot|_2$ to $|\cdot|_G$ and to $|\cdot|_{H^2}$ on H_1^2.

2) $k = 3$. The proof is similar to the previous case $(k = 1)$, once we observe that $u \in H_3^2$ if and only if there exists a $v \in H^0$ such that $D^2u = v$, $u(0) = u'(0) = 0$, namely $u = \int_0^x \int_0^{s_1} v(s)dsds_1$.

The domains of the operators we will study will be certain dense subsets of H_k^2 satisfying particular boundary conditions.

Definition 1.9. We say ϕ satisfies boundary conditions of type k, $k = 1,2$, or 3, if

a) $\phi(0) = \phi''(0) = \phi(1) = \phi''(1) = 0$ for $k = 1$,

b) $\phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0$ for $k = 2$,

c) $\phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0$ for $k = 3$.

Define $H_k^4 = \{\phi \in H^4 : \phi$ satisfies boundary conditions of type $k\}$.

Note that for the Euler-Bernoulli beam, a) corresponds to the boundary conditions for the simply supported beam, b) to boundary conditions for a beam clamped at both ends, and c) to boundary conditions for a cantilevered beam.

We will also need to use the following fact:

$$H_k^4 \text{ is dense in } H^0 \text{ (in the } |\cdot|_0 \text{ norm).}$$

This follows from the fact that H_0^m can be characterized as the closure of $C_0^\infty(0,1)$ in the H^m norm [1, p. 44; 30, p. 91], and that $C_0^\infty \subset H_0^m \subset H_k^4 \subset H^0$ for $m \geq 4$; the result follows since C_0^∞ is dense in H^0 [1, p. 31].

Section 3. Splines and Error Bounds for Approximation by Splines

The approximating subspaces $Z^N \subset Z$ will consist of sets of cubic or quintic splines satisfying prescribed boundary conditions. We begin this section by defining a particular class of interpolating functions, the L-splines.

Given any partition $\Delta^N = \{x_i\}_{i=0}^N$, $0 = x_0 < x_1 < \ldots < x_N = 1$, the L-splines, as defined by Schultz and Varga [41], are piecewise continuous functions satisfying certain interpolating conditions. The functions themselves belong to the kernel of certain differential operators on each $[x_i, x_{i+1}]$. To be more precise, given any differential operator L of order m

$$(1.12) \qquad Lu(x) = \sum_{j=0}^{m} a_j(x) D^j u(x), \quad m \geq 1,$$

where $a_j \in C^j[a,b]$, and given any partition Δ^N, we define an incidence vector $d = (d_1, \ldots, d_{N-1})$ of positive integers with $1 \leq d_i \leq m$ for $1 \leq i \leq N-1$, and define the L-spline space in the following way:

Definition 1.10. [41]. The L-spline space $Sp(L, \Delta^N, d)$ is the collection of all functions on $[a,b]$ such that

$$L^*Ls(x) = 0, \quad x \in [a,b] \setminus \Delta^N$$

$$D^k s(x_i^-) = D^k s(x_i^+) \quad \text{for all } 0 \leq k \leq 2m-1-d_i, \quad 0 < i < N,$$

where L^* is the formal adjoint of L.

We will be interested in the specific case where $L = D^m$ and $d_i = 1$, $0 < i < N$. In this case, the functions in $Sp(L, \Delta^N, d)$ are piecewise polynomials of degree $2m-1$, with $C^{2m-2}[0,1]$ continuity. In order to obtain error estimates for projections onto spaces of splines, we use well-known error estimates for interpolation by L-splines. For our purposes, we require the type-1 interpolant of [41].

Definition 1.11. [41]. Given $f(x) \in C^{m-1}[a,b]$ and with L the differential operator of order m, Δ^N, and d as above, we say a function $s(x) \in Sp(L, \Delta^N, d)$ is a $Sp(L, \Delta^N, d)$-interpolant of $f(x)$ of type 1 if

i) $s(x_i) = f(x_i)$, $0 < i < N$

ii) $(D^k s)(x_i) = (D^k f)(x_i)$, $0 \leq k \leq m-1$ for $i = 0$ and $i = N$.

It is known [41] that, given any $f \in C^{2m-1}[0,1]$, there exists a unique $s \in Sp(L,\Delta^N,d)$ such that s is a type-1 interpolant of f.

For each positive integer N, let $\Delta^N = \{x_i\}_{i=0}^N$, $x_i = i/N$; in the sequel, we will restrict our attention to such uniform partitions of $[0,1]$. Also we take $L \equiv D^m$ and $d_i = 1$, $i = 1, \ldots, N-1$. The corresponding spline subspaces will be denoted $S^{2m-1}(\Delta^N) \equiv Sp(D^m,\Delta^N,d)$. Thus a spline of degree $2m-1$ (order $2m$) will be a function $s(x) \in S^{2m-1}(\Delta^N) \subset C^{2m-2}$ such that $s(x)$ is a polynomial of degree $2m-1$ (order $2m$) on each subinterval $[x_i,x_{i+1}]$ defined by the partition Δ^N. In particular,

$$S^3(\Delta^N) = \{s(x) \in C^2[0,1]: s \text{ is a polynomial of degree } 3 \text{ on } [x_i,x_{i+1}],\ i = 0,N-1\}$$

is the set of <u>cubic splines</u> $(m = 2)$ and

$$S^5(\Delta^N) = \{s(x) \in C^4[0,1]: s \text{ is a polynomial of degree } 5 \text{ on } [x_i,x_{i+1}],\ i = 0,N-1\}$$

is the set of <u>quintic splines</u> $(m = 3)$. The type-1 interpolant s to $f \in C^1[0,1]$ from $S^3(\Delta^N)$ satisfies

i) $s(x_i) = f(x_i)$, $0 < i < N$,

and

ii) $s(0) = f(0)$, $s(1) = f(1)$

$(Ds)(0) = (Df)(0)$, $(Ds)(1) = (Df)(1)$.

Given $f \in C^2[0,1]$, the type-1 interpolant s to f from $S^5(\Delta^N)$

satisfies

i) $s(x_i) = f(x_i), \quad 0 < i < N$

ii) $s(0) = f(0), \quad s(1) = f(1)$

$(Ds)(0) = (Df)(0), \quad (Ds)(1) = (Df)(1)$

$(D^2s)(0) = (D^2f)(0), \quad (D^2s)(1) = (D^2f)(1).$

These finite-dimensional subspaces $S^{2m-1}(\Delta^N)$ of H^{2m-1} possess a convenient basis (the B-splines) which will be defined at the end of this section. Suffice it to say for the moment that the B-spline basis functions have small (local) support which make the spaces $S^{2m-1}(\Delta^N)$ efficient for Galerkin-type approximations. The second advantage of splines is their approximation power.

Since we shall be approximating functions using projections of the form $P^N: H^k \to S^{2m-1}(\Delta^N)$, $k \in \{0,\ldots,m-1\}$, we shall need to obtain error bounds for such approximations by splines. The following result for bounding the error in spline interpolation will be fundamental in deriving error bounds for projections onto subspaces of splines. While the results of this theorem have been proved for general partitions (satisfying a uniformity condition $\max\limits_{0\leq i\leq N-1} (x_{i+1}-x_i)/\min\limits_{0\leq i\leq N-1} (x_{i+1}-x_i) \leq \sigma$, some $\sigma \geq 1$) and for the general L in (1.12), we state the results only for the particular case we need.

Theorem 1.13. [44]. Given $f \in H^{2m}$, let s be the unique element in $S^{2m-1}(\Delta^N)$ which interpolates f in the sense of a type-1 interpolant. Then,

$$|D^j(f-s)|_0 \leq c_{j,m}\left(\frac{1}{N}\right)^{2m-j}|D^{2m}f|_0, \quad j = 0,\ldots,2m-1,$$

where $c_{j,m}$ are constants independent of f.

Remark. This result is proved for $j = 0,\ldots,m$ in [41, Theorem 9]. For the case where $m+1 \le j \le 2m-1$, see [44, Lemma 3.1] and the references there. We will also need the Schmidt inequality [40, p. 7]:

Lemma 1.14. If $p_n(x)$ is a polynomial of degree $n = 1,2, \ldots,5$, then $\int_a^b [Dp_n(x)]^2 dx \le \tilde{c}_n (b-a)^{-2} \int_a^b [p_n(x)]^2 dx$, where $\tilde{c}_1 = 12$, $\tilde{c}_2 = 60$, $\tilde{c}_3 = 2(45 + \sqrt{1605}) \approx 170.$, $\tilde{c}_4 \approx 440$, $\tilde{c}_5 \approx 738.8$.

Now we wish to obtain error bounds for projections onto certain subspaces of $S^3(\Delta^N)$ and $S^5(\Delta^N)$ consisting of splines satisfying prescribed boundary conditions. In [7], we proved the needed results for a certain subspace of cubic splines, and we restate those results here.

Denote by $S_0^3(\Delta^N) = \{s \in S^3(\Delta^N)\colon s(0) = s(1) = 0\}$. Given a function $z \in H^0$, denote by $P_0^N z$ its projection onto $S_0^3(\Delta^N)$ in the H^0 norm. P_0^N is the map $P_0^N\colon H^0 \rightarrow S_0^3(\Delta^N)$ satisfying $\langle z - P_0^N z, s\rangle_0 = 0$ for every $s \in S_0^3(\Delta^N)$, or $|P_0^N z - z|_0 = \inf\{|s-z|_0 \mid s \in S_0^3(\Delta^N)\}$. In [7], we prove

Lemma 1.15. [7, 2.3]. If $z \in H_0^1 \cap H^4 \subset H^0$, and P_0^N is the projection $P_0^N\colon H^0 \longrightarrow S_0^3(\Delta^N)$, then

$$|D^j(P_0^N z - z)|_0 \le \kappa_{0,j} \left(\tfrac{1}{N}\right)^{4-j} |D^4 z|_0, \quad j = 0,1,2.$$

Take $H_0^1(\alpha)$ to be, as defined in Section 2, H_0^1 equipped with the $|\cdot|_{1,\alpha}$ norm induced by the inner product $\langle\phi,\psi\rangle_{1,\alpha} = \langle\alpha D\phi, D\psi\rangle_0$. Given a function $z \in H_0^1$, denote by $P_1^N z$ its projection onto the subspace $S_0^3(\Delta^N)$. P_1^N is the map $P_1^N\colon H_0^1(\alpha) \longrightarrow S_0^3(\Delta^N)$

satisfying $<z - P_1^N z, s>_{1,\alpha} = 0$ for every $s \in S_0^3(\Delta^N)$, or $|P_1^N z - z|_{1,\alpha} = \inf\{|s-z|_{1,\alpha} \mid s \in S_0^3(\Delta^N)\}$. In [7], we also prove

Lemma 1.16. [7, 4.3]. If $z \in H_0^1 \cap H^4$ and P_1^N is the projection $P_1^N\colon H_0^1 \longrightarrow S_0^3(\Delta^N)$ in the $|\cdot|_{1,\alpha}$ norm, then

$$|D^j(P_1^N z - z)|_{1,\alpha} \leq \sqrt{\alpha}\, \kappa_{1,j} (\tfrac{1}{N})^{3-j} |D^4 z|_0, \quad j = 0,1,2.$$

Note that P_0^N and P_1^N are both orthogonal projections onto $S_0^3(\Delta^N)$, but the projections are taken with respect to different norms. Also, since H^4 is dense in H^1 and in H^0, the above error bounds hold on dense subsets $(H^4 \cap H_0^1)$ of the domain of definition of the projection operators.

We wish to state and prove analogous results for projections onto spaces of quintic splines satisfying prescribed boundary conditions. Denote by $S_k^5(\Delta^N)$ the collection of quintic splines satisfying boundary conditions of type k, defined in Section 2:

$$\begin{aligned} S_k^5(\Delta^N) &= \{s \in S^5(\Delta^N)\colon s \text{ satisfies boundary conditions of type } k,\ k = 1,2,3\}, \\ &= \{s \in C^4[0,1]\colon s \text{ is a quintic polynomial on each } [x_i, x_{i+1}],\ i = 0,\ldots,N-1, \text{ and } s \text{ satisfies the boundary conditions of type } k,\ k = 1,2,3\}. \end{aligned}$$

We shall again require two sets of projections. Those we need are

$$P_{2,k}^N\colon H^0 \longrightarrow S_k^5(\Delta^N)$$

and

$$P_{3,k}^N\colon H_k^2(\alpha) \longrightarrow S_k^5(\Delta^N).$$

First, we obtain some preliminary bounds using the Schmidt inequality.

<u>Lemma 1.17</u>. Let $s \in S_k^5(\Delta^N)$ be a quintic spline satisfying boundary conditions fo type 1, 2 or 3. Then

i) $|D^j s|_0 \le \hat{c}_j N^j |s|_0$, $1 \le j \le 4$

ii) $|D^2 s|_{2,\alpha} \le \hat{c}_5 N^2 |s|_{2,\alpha}$,

and

iii) $|s|_{2,\alpha} \le \sqrt{\alpha}\ \hat{c}_6 N^2 |s|_0$.

<u>Proof</u>: Apply the Schmidt inequality:

$$\int_0^1 |Ds(x)|^2 dx = \sum_{i=1}^N \int_{x_{i-1}}^{x_i} |Ds(x)|^2 \le \sum_{i=1}^N \tilde{c}_5 N^2 \int_{x_{i-1}}^{x_i} |s(x)|^2,$$

since $s(x)$ is a polynomial of degree 5 on $[x_{i-1},x_i]$, $i = 1, \ldots, N$. This implies i) with $\hat{c}_1 = \sqrt{\tilde{c}_5}$, $j = 1$. Let $g(x) = Ds(x)$. Then g is a polynomial of degree 4 on each $[x_{i-1},x_i]$, $i = 1, \ldots, N$, and

$$\begin{aligned}
\int_0^1 |D^2 s(x)|^2 dx = \int_0^1 |Dg(x)|^2 dx &\le \sum_{i=1}^N \tilde{c}_4 N^2 \int_{x_{i-1}}^{x_i} |g(x)|^2 dx \\
&= \tilde{c}_4 N^2 \int_0^1 |g(x)|^2 dx \\
&= \tilde{c}_4 N^2 \int_0^1 |Ds(x)|^2 dx \\
&\le \tilde{c}_4 \hat{c}_1^2 N^4 \int_0^1 |s(x)|^2 dx
\end{aligned}$$

which establishes (i) with $\hat{c}_2 = \sqrt{\tilde{c}_4\ \tilde{c}_5}$, $j = 2$. In other cases ($j = 3,4$) follow in a similar manner. Inequality (ii) is derived in the same manner. Let $g(x) = D^2 s(x)$; then g is a cubic polynomial on $[x_{i-1},x_i]$, $i = 1, \ldots, N$, and

$$|D^2 s|^2_{2,\alpha} = \alpha |D^2 g|^2_0 = \alpha \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} D^2 g(x)dx$$

$$= \alpha \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} D(Dg(x))dx$$

$$\leq \alpha\ \tilde{c}_2 N^2 \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} Dg(x)dx$$

$$\leq \alpha\ \tilde{c}_2 \tilde{c}_3\ N^4 \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} g(x)dx$$

$$= \alpha\ \tilde{c}_2 \tilde{c}_3 N^4 |g|^2_0$$

$$= \tilde{c}_2 \tilde{c}_3 N^4 |s|^2_2 ,$$

which proves (ii) with $\hat{c}_5 = \sqrt{\tilde{c}_2 \tilde{c}_3}$. Finally

$$|s|^2_{2,\alpha} = \alpha |D^2 s|^2_0 = \alpha \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} D(Ds(x))dx$$

$$\leq \alpha\ \tilde{c}_4 N^2 \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} Ds(x)dx$$

$$\leq \alpha\ \tilde{c}_4 \tilde{c}_5 N^4 \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} s(x)dx$$

$$= \alpha\ \tilde{c}_4 \tilde{c}_5 N^4 |s|^2_0,$$

which establishes (iii) with $\hat{c}_6 = \sqrt{\tilde{c}_4 \tilde{c}_5}$. □

Finally, we bound the error for projections onto $S^5_k(\Delta^N)$ for $k = 1,2$.

Lemma 1.18. Let $P^N_{2,k}$ be the orthogonal projection $P^N_{2,k}$: $H^0 \longrightarrow S^5_k(\Delta^N)$ with respect to the H^0 norm, for $k = 1$ or 2; then if $z \in \{\phi \in H^6 : \phi$ satisfies boundary conditions of type $k\}$

i) $|P^N_{2,k} z - z|_0 \leq \kappa_{2,0} (\frac{1}{N})^6\ |D^6 z|_0$

ii) $|D^j(P^N_{2,k}z-z)|_0 \leq \kappa_{2,j}(\frac{1}{N})^{6-j}|D^6z|_0, \quad j = 1,\ldots,5,$

and

iii) $|P^N_{2,k}z-z|_{2,\alpha} \leq \sqrt{\alpha}\ \kappa_{2,2}(\frac{1}{N})^4|D^6z|_0.$

Proof: The first estimate (i) follows directly from

$$|P^N_{2,k}z-z|_0 \leq |I^N_kz-z|_0$$

where I^N_kz is the type-1 interpolant to z from $S^5(\Delta^N)$. Since z satisfies the boundary conditions of type k, and type-1 interpolants interpolate these conditions, it follows that $I^N_kz \in S^5_k(\Delta^N)$. Furthermore, the basic spline interpolation error bounds (Theorem 1.13) hold, and thus (i) follows directly.

To obtain (ii), we write

$$|D^j(P^N_{2,k}z-z)|_0 \leq |D^j(P^N_{2,k}-I^N_kz)|_0 + |D^j(I^N_kz-z)|_0. \tag{1.13}$$

A bound on the second term in (1.13) follows directly from the interpolation error bounds:

$$|D^j(I^N_kz-z)|_0 \leq c_{j,3}(\frac{1}{N})^{6-j}|D^6z|_0.$$

To bound the first term in (1.13), we again resort to the Schmidt inequality, since $P^N_{2,k}z - I^N_kz \in S^5_k(\Delta^N)$,

$$\begin{aligned}|D^j(P^N_{2,k}z-I^N_kz)|_0 &\leq \hat{c}_jN^j|P^N_{2,k}z-I^N_kz|_0, \quad \text{by Lemma 1.17,}\\ &\leq 2\hat{c}_jN^j|I^N_kz-z|_0\\ &\leq 2\hat{c}_jN^{j-6}\ c_{0,3}|D^6z|_0, \text{ by Lemma 1.13,}\end{aligned}$$

which gives us (ii) with $\kappa_{2,j} = 2\hat{c}_jc_{0,3} + c_{j,3}$. Finally

$$|P^N_{2,k}z-z|_{2,\alpha} = \sqrt{\alpha}|D^2(P^N_{2,k}z-z)|_0 \le \sqrt{\alpha}\ \kappa_{2,2}(\tfrac{1}{N})^4|D^6z|_0. \quad \square$$

The final projection error bounds which we shall require are those for the projections in the $H^2_k(\alpha)$ norm. Recall that $H^2_k = \{\phi \in H^2$: ϕ satisfies boundary conditions of type k, $k = 1,2$, or $3\}$, and that $H^2_k(\alpha)$ is H^2_k equipped with the norm $|\cdot|_{2,\alpha}$ derived from the inner product $\langle\cdot,\cdot\rangle_{2,\alpha} \equiv \langle\alpha D^2\cdot,D^2\cdot\rangle_0$. We define $P^N_{3,k}$ to be the orthogonal projection $P^N_{3,k}$: $H^2_k(\alpha) \longrightarrow S^5_k(\Delta^N)$ satisfying

$$|P^N_{3,k}z-z|_{2,\alpha} = \inf\{|s-z|_{2,\alpha} \mid s \in S^5_k(\Delta^N)\}.$$

Lemma 1.19. Let $P^N_{3,k}$ be the projection $P^N_{3,k}$: $H^2_k(\alpha) \longrightarrow S^5_k(\Delta^N)$ taken with respect to the $|\cdot|_{2,\alpha}$ norm. If $z \in \{\phi \in H^6$: ϕ satisfies boundary conditions of type $k\}$ and $k = 1$ or 2, then

i) $|P^N_{3,k}z-z|_{2,\alpha} \le \sqrt{\alpha}\ \kappa_{3,0}(\frac{1}{N})^4|D^6z|_0$

ii) $|D^2(P^N_{3,k}z-z)|_{2,\alpha} \le \sqrt{\alpha}\ \kappa_{3,2}(\frac{1}{N})^2|D^6z|_0$

iii) $|D^4(P^N_{3,k}z-z)|_0 \le \kappa_{3,2}(\frac{1}{N})^2|D^6z|_0$.

Proof: Let I^N_kz be the type-1 interpolant from $S^5(\Delta^N)$ to z. Then since type-1 interpolants interpolate the boundary conditions of type k (for $k = 1$ or 2), $I^N_kz \in S^5_k(\Delta^N)$, and the spline interpolation error bounds of Theorem 1.13 hold. The inequality (i) follows directly from

$$|P^N_{3,k}z-z|_{2,\alpha} \le |I^N_kz-z|_{2,\alpha} = \sqrt{\alpha}|D^2(I^N_kz-z)|_0 \le \sqrt{\alpha}\ c_{2,3}(\tfrac{1}{N})^4|D^6z|_0.$$

To obtain (ii), write

(1.14) $$|D^2(P^N_{3,k}z-z)|_{2,\alpha} \le |D^2(P^N_{3,k}z-I^N_kz)|_{2,\alpha} + |D^2(I^N_kz-z)|_{2,\alpha}.$$

The second term of (1.14) satisfies

$$|D^2(I^N_kz-z)|_{2,\alpha} = \sqrt{\alpha}|D^4(I^N_kz-z)|_0 \le \sqrt{\alpha}\ c_{4,3}(\tfrac{1}{N})^2|D^6z|_0$$

from the spline interpolation error bounds (Lemma 1.13).

To bound the second term we again use the Schmidt inequality and the fact that $P^N_{3,k}z - I^N_kz \in S^5_k(\Delta^N)$:

$$\begin{aligned}|D^2(P^N_{3,k}z-I^N_kz)|_{2,\alpha} &\le \hat{c}_5N^2|P^N_{3,k}z-I^N_kz|_{2,\alpha}, \text{ from Lemma 1.14,}\\ &\le \hat{c}_5N^2(|P^N_{3,k}z-z|_{2,\alpha} + |z-I^N_kz|_{2,\alpha})\\ &\le 2\hat{c}_5N^2|z-I^N_kz|_{2,\alpha}\\ &= 2\ \sqrt{\alpha}\ \hat{c}_5N^2\ |D^2(I^N_kz-z)|_0\\ &= 2\ \sqrt{\alpha}\ \hat{c}_5(\tfrac{1}{N})^2\ c_{2,3}|D^6z|_0\\ &= \sqrt{\alpha}\ \kappa_{3,2}(\tfrac{1}{N})^2|D^6z|_0,\end{aligned}$$

with $\kappa_{3,2} = c_{4,3} + 2\hat{c}_5c_{2,3}$. □

<u>Remark</u>. Projection error bounds for projections onto splines satisfying cantilever-type boundary conditions (type 3) have been excluded from the above. The reason for this is that they do not fit into the above framework since the type-1 interpolant does not interpolate these conditions. Furthermore, the proof of the interpolation error bounds requires that the interpolating spline to f satisfy the first integral relation (cf: [48])

$$\int_0^1(Lf)^2dx = \int_0^1\{L(f-s)\}^2 + \int_0^1(Ls)^2dx,$$

where $L = D^3$ here. For this to hold, it is necessary that

$$\int_0^1 (D^3 s)^2 dx = (-1)\int_0^1 s(D^6 s) dx.$$

But the integration by parts fails for type 3 boundary conditions. However, our numerical results suggest that the projections $P^N_{2,k}$ and $P^N_{3,k}$ when $k = 3$ satisfy error bounds similar to those for $P^N_{2,k}$ and $P^N_{3,k}$ when $k = 1,2$.

In addition to their strong approximation properties, the spaces $S^{2m-1}(\Delta^N)$ are attractive computationally because they possess a computationally simple basis of small (local) support, namely the B-splines. The B-splines are compactly defined as differences of the truncated power basis functions.

All the B-splines are translates of one basic B-spline. Define $\overline{B}_n(x) = \delta^n (x-y)^{n-1}_+ = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^{n-1}_+$, as the basic B-spline with knots at $0,1,\ldots,n$, where δ^n is the forward difference operator $\delta f(x_0) = f(x_1) - f(x_0),\ldots,\delta^{k+1} f(x_0) = \delta^k f(x_1) - \delta^k f(x_0)$, and where

$$(x-t)^{n-1}_+ = \begin{cases} (x-t)^{n-1}, & \text{when } t \le x \\ 0 & \text{, when } t > x \end{cases}$$

is the truncated power basis function.

To define the B-spline basis for $S^{2m-1}(\Delta^N)$, take $n = 2m$, and define

$$\hat{B}^N_{i,n-1}(x) = \overline{B}((x-y_i)/h)$$

where $h = x_{i+1} - x_i$ and where we have defined an extended partition $\overline{\Delta}^N = \{x_i\}_{i=1-m}^{N+m-1}$, $x_i = ih$, of Δ^N, and $y_i = x_i - mh$.

Remark. With the B-splines defined in this manner, $B^N_{i,3}$ agrees with the Prenter definition of cubic splines [35]. Its relation to the normalized B-splines of DeBoor (cf: [42, p. 135]) is given by $B^N_{i,n} = \frac{1}{(m-1)!} N^n_i(x)$.

The support of each B-spline $\hat{B}_{i,2m-1}$ is the interval $[x_{i-m},x_{i+m}]$, and it is easily shown [42, p.116] that the $N+m+1$ B-splines $\{\hat{B}^N_{i,2m-1}\}^{N+m-1}_{i=1-m}$ span $S^{2m-1}(\Delta^N)$. Moreover, since all B-splines are obtained by scaling and translating one basic B-spline, it is efficient to store a function $s \in S^{2m-1}(\Delta^N)$ by storing its coefficients: $s \in S^{2m-1}(\Delta^N)$ implies $s(x) = \sum_{i=1-m}^{N+m-1} c_i B^N_{i,2m-1}(x)$. The computational aspects of B-splines will be explored in Chapter 4.

Finally, recall that we require approximating subspaces in $\mathrm{Dom}(\mathscr{A})$. Thus we will require our splines to satisfy the boundary conditions. For the cubic splines in $S^3_0(\Delta^N)$, we require $s(0) = s(1) = 0$. We may take from the $N+3$ basis elements in $S^3(\Delta^N)$ the following $N+1$ basis elements for $S^3_0(\Delta^N)$:

$$
\begin{aligned}
& B^N_{i,3} = \hat{B}^N_{i,3}, \qquad i = 2,\ldots,N-2 \\
(1.15)\quad & B^N_{0,3} = \hat{B}^N_{0,3} - 4\hat{B}_{-1,3}; \qquad B^N_{1,3} = \hat{B}_{1,3} - \hat{B}_{0,3}/4 \\
& B^N_{N-1,3} = \hat{B}_{N-1,3} - \hat{B}_{N,3}/4; \qquad B^N_{N,3} = \hat{B}^N_{N,3} - 4\hat{B}^N_{N+1,3}.
\end{aligned}
$$

The same approach can be used to obtain a basis for quintic splines satisfying boundary conditions of type k. These are listed as follows:

i) To obtain a basis for $S^5_1(\Delta^N)$, take from $\{B^N_{i,5}\}^{N+2}_{i=-2}$ the following $N+1$ splines $\{B^N_i\}^N_{i=0}$ (dropping the second subscript)

$$
(1.16)\qquad
\begin{aligned}
B_i^N &= \hat{B}_i^N, \quad 3 \le i \le N-3 \\
B_0^N &= \hat{B}_0^N - 3\hat{B}_{-1}^N + 12\hat{B}_{-2}^N \\
B_1^N &= \hat{B}_1^N - \hat{B}_{-1}^N \\
B_2^N &= \hat{B}_2^N - \hat{B}_{-2}^N
\end{aligned}
$$

with $B_{N-j}(x) = B_j(1-x)$, $j = 0,1,2.$

ii) To obtain a basis for $S_2^5(\Delta^N)$, take

$$
(1.17)\qquad
\begin{aligned}
B_0^N &= \hat{B}_0^N - \frac{33}{8}\hat{B}_{-1}^N + \frac{165}{4}\hat{B}_{-2}^N \\
B_1^N &= \hat{B}_1^N - \frac{9}{4}\hat{B}_{-1}^N + \frac{65}{2}\hat{B}_{-2}^N \\
B_2^N &= \hat{B}_2^N - \frac{1}{8}\hat{B}_{-1}^N + \frac{9}{4}\hat{B}_{-2}^N
\end{aligned}
$$

with $B_j^N = \hat{B}_j^N$, $3 \le j \le N-3$ and $B_{N-j}^N(x) = B_j^N(1-x)$, $0 \le j \le 2.$

iii) For the cantilevered beam, we obtain a basis for $S_3^5(\Delta^N)$ by taking $B_0^N, \dots, B_{N-3}^N$ as in ii) and

$$
(1.18)\qquad
\begin{aligned}
B_N^N &= \hat{B}_N^N + \frac{3}{2}\hat{B}_{N+1}^N + 3\hat{B}_{N+2}^N \\
B_{N-1}^N &= \hat{B}_{N-1}^N - 2\hat{B}_{N+2}^N \\
B_{N-2}^N &= \hat{B}_{N-2}^N - \frac{1}{2}\hat{B}_{N+1}^N
\end{aligned}
$$

Having defined the IDA problem and its approximation IDN, and having defined the spline spaces necessary to set up the approximating subspaces Z^N, we now proceed to some concrete applications of these ideas.

CHAPTER 2. APPLICATION TO THE EULER-BERNOULLI EQUATION

Section 1. The Euler-Bernoulli Equation with Structural and Viscous Damping

Having introduced the basic ideas for the parameter estimation problem, we turn to the application of these ideas to specific equations arising in elasticity. In this chapter, we discuss the Euler-Bernoulli equations. The well-known equations for the transverse vibrations of a thin elastic beam are

$$\mathcal{M} = EI\,\frac{\partial^2 y}{\partial x^2}$$

$$m\,\frac{\partial^2 y}{\partial t^2} + \frac{\partial^2 \mathcal{M}}{\partial x^2} = f(t,x;q) \tag{2.1}$$

where $\mathcal{M}$ is the bending moment, m is mass per unit length and f is the applied load. We wish to include two types of damping in the above model. The first is velocity-proportional viscous damping γy_t. The second is structural damping arising from a simple viscoelastic model, the Kelvin-Voight model, where we have the constitutive relationship $\sigma = E\varepsilon + c\dot{\varepsilon}$, where σ,ε are the linear stress and strain, E is Young's modulus, and c is the damping coefficient. Thus we introduce damping proportional to strain velocity.

Following the usual development of the Euler-Bernoulli equation [14, pp. 295-302], we obtain

$$\mathcal{M} = \int \sigma y\,dA = EI\,\frac{\partial^2 y}{\partial x^2} + cI\,\frac{\partial^3 y}{\partial x^2 \partial t}$$

and the equation (2.1) becomes

$$(2.2) \qquad y_{tt} = -\alpha D^4 y - \delta D^4 y_t - \gamma y_t + f, \quad t > 0, \quad x \in [0,1],$$

where $\alpha > 0$, $\delta \geq 0$, $\gamma \geq 0$.

In setting up the operators and appropriate spaces below, we shall consider only homogeneous boundary conditions of type k introduced in Definition 1.9. This results in no loss of generality of the methods, since any non-homogeneous conditions, including time dependent conditions (which can occur due to applied moments and shears at the boundary) may be transformed (see [29]) in a manner that reduces the equations to ones with homogeneous boundary conditions and an additional "load" term included in f.

Within the framework we have established, a variety of approximations may be used. Recall that we rewrite the boundary-initial value problem as the abstract equation

$$\begin{aligned} \dot{z}(t) &= \mathscr{A}(q)z(t) + F(t;q) \quad \text{on} \quad Z \\ z(0) &= z_0 \end{aligned}$$

and then approximate it by a sequence of equations of the form

$$\begin{aligned} \dot{z}^N(t) &= \mathscr{A}^N(q)z^N(t) + F^N(t;q) \quad \text{on} \quad Z^N \\ z^N(0) &= z_0^N. \end{aligned}$$

One way of obtaining different approximations clearly is by making different choices for $\mathscr{A}^N$. We discuss only the choice $\mathscr{A}^N = P^N \mathscr{A} P^N$ here, but other choices are possible. For example, if $\mathscr{A}^{1/2}$ exists as a differential operator, then $P^N \mathscr{A}^{1/2} P^N \mathscr{A}^{1/2} P^N$ can be used (see [27]).

Additionally, the choice of state spaces and the form of the evolution equation on these spaces leads to different natural approximations; it is this difference we wish to examine here.

In particular, for the equation (2.2), we shall write

$$(2.3)\qquad \begin{cases} \dot z(t) = \mathscr{A}(q)z(t) + F(q,t) \\ z(0) = z_0 \\ \mathscr{A}(q) = \begin{pmatrix} 0 & 1 \\ -q_1D^4 & -q_2D^4-q_3 \end{pmatrix} \\ \text{in} \quad X = H^2_k(\alpha) \times H^0 \end{cases}$$

or, (when $\gamma = 0$)

$$(2.4)\qquad \begin{cases} \Gamma(q)\ \dot z(t) = \mathscr{A}(q)z(t) + F(q,t) \\ \qquad z(0) = z_0 \\ \mathscr{A}(q) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -q_1D^2 \\ 0 & D^2 & 0 \end{pmatrix} \\ \Gamma(q) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & q_2D^2 \\ 0 & 0 & 1 \end{pmatrix} \\ \text{in} \quad X = H^0 \times H^0 \times H^0, \end{cases}$$

where $q_1 = \alpha$, $q_2 = \beta$, $q_3 = \gamma$.

As we shall see, (2.3) leads naturally to quintic spline approximations which are discussed in section (2) and (2.4) leads to cubic spline approximations which are discussed in section (3).

Section 2. An Approximation Using Quintic Splines

In order to investigate and approximate solutions of

$$(2.4)\qquad \begin{cases} y_{tt} = -\alpha y_{xxxx} - \delta y_{xxxxt} + f \\ y(0,x) = \phi(x) \\ y_t(0,x) = \psi(x) \end{cases}$$

where y satisfies boundary conditions of type k, we write (2.4) as abstract equation in a subspace of $H^2 \times H^0$

$$(2.5)\qquad \begin{aligned} &\dot{z}(t) = \mathscr{A}(q)z(t) + F(q,t), \quad t > 0 \\ &z(0) = z_0 \end{aligned}$$

with $z(t) = (z_1(t),z_2(t))^T \equiv (y(t,\cdot),y_t(t,\cdot))^T$, $z_0 = (\phi,\psi)^T$, and $F(q,t) = (0,f(t,\cdot;q))^T$, and

$$\mathscr{A}(q) = \begin{bmatrix} 0 & 1 \\ -q_1D^4 & -q_2D^4 \end{bmatrix}$$

where $q_1 \equiv \alpha$, $q_2 \equiv \delta$. In particular, for each of the boundary conditions of type k, $k = 1,2,3$, we consider (2.5) in $H_k^2 \times H^0$, and $\mathscr{A}_k$ to be of the form $\mathscr{A}$, with $\mathrm{Dom}(\mathscr{A}_k) = H_k^4 \times H_k^4$.

We derive the important properties of these operators which will be used to prove convergence of the approximate identification problem associated with (2.5).

We first consider the special case where $\delta = 0$ (no damping) and denote the corresponding operators by $\hat{\mathscr{A}}_k$ (i.e., $\hat{\mathscr{A}}_k$ is $\mathscr{A}_k$ in case $q_2 = 0$). In this case, we can define the operators $\hat{\mathscr{A}}_k$ in such a way that maximal dissipativeness can easily be argued in the following way: we will show that $\hat{\mathscr{A}}_i$ and $\hat{\mathscr{A}}_i^*$ are disspative, and $\hat{\mathscr{A}}_i$ is closed; maximal dissipativeness then follows [25, Theorem 4.4, p. 87].

To extend the results to the case $\delta > 0$, we will also need to prove dense inclusions of certain subsets of Sobolev spaces. To do this, we will use the following lemma:

Lemma 2.1. Let $\mathscr{A}: (\mathrm{Dom}(\mathscr{A}) \subset X) \to X$ be linear, X a Hilbert space. If $\mathscr{A}$ satisfies the dissipative inequality $\langle \mathscr{A}x,x\rangle \leq 0$ for every $x \in \mathrm{Dom}(\mathscr{A})$ and if $R(\mathscr{A} - \lambda I) = X$ for some $\lambda > 0$, then $\mathrm{Dom}(\mathscr{A})$ is dense in X.

Proof: Suppose $\mathrm{Dom}(\mathscr{A})$ is not dense. Then, there exists a non-zero $x_0 \in X$ such that $\langle x_0,x\rangle = 0$ for every $x \in \mathrm{Dom}(\mathscr{A})$. Since by assumption $R(\mathscr{A} - \lambda I) = X$ for some $\lambda > 0$, it follows that $x_0 = (\mathscr{A} - \lambda I)y_0$ for some non-zero $y_0 \in \mathrm{Dom}(\mathscr{A})$. Therefore,

$$\begin{aligned} 0 &= \langle x_0,x\rangle \quad \text{for every } x \in \mathrm{Dom}(\mathscr{A}) \\ &= \langle x_0,y_0\rangle, \text{ in particular,} \\ &= \langle(\mathscr{A} - \lambda I)y_0,y_0\rangle \\ &= \langle \mathscr{A}y_0,y_0\rangle - \lambda|y_0|^2 \\ &< \langle \mathscr{A}y_0,y_0\rangle, \end{aligned}$$

which contradicts the dissipative inequality on $\mathscr{A}$. □

Corollary 2.2. Let $\mathscr{A}: (\mathrm{Dom}(\mathscr{A}) \subset X) \to X$ be linear, X a Hilbert space. If $\mathscr{A}$ satisfies a dissipative inequality $\langle \mathscr{A}x,x\rangle \leq 0$ for every $x \in \mathrm{Dom}(\mathscr{A})$ and if $R(\mathscr{A}) = X$, then $\mathrm{Dom}(\mathscr{A})$ is dense in X.

Proof: $R(\mathscr{A}) = X$ implies $0 \in \rho(\mathscr{A})$, the resolvent set of $\mathscr{A}$. But $\rho(\mathscr{A})$ is an open set, and thus there exists a $\lambda > 0$ such that $\lambda \in \rho(\mathscr{A})$ and $R(\mathscr{A} - \lambda I) = X$. The above theorem now yields the result. ▫

Finally, we will obtain the desired results for the case $\delta > 0$ by taking maximal dissipative extensions of $\mathscr{A}_k$. We consider the six cases corresponding to $\hat{\mathscr{A}}_k$, $k = 1,2,3$, and $\mathscr{A}_k$, $k = 1,2,3$.

Case 1. Consider the case corresponding to a simply supported beam. Consider the operator in $Z = H_1^2(\alpha) \times H^0$

$$\hat{\mathscr{A}}_1 = \begin{pmatrix} 0 & 1 \\ -\alpha D^4 & 0 \end{pmatrix} \quad \text{on} \quad \mathrm{Dom}(\hat{\mathscr{A}}_1) = H_1^4 \times H_1^2$$

where

$$H_1^4 = \{\phi \in H^4 : \phi(0) = \phi''(0) = \phi(1) = \phi''(1) = 0\}$$

$$H_1^2 = \{\phi \in H^2 : \phi(0) = \phi(1) = 0\}$$

and $H_1^2(\alpha) = H_1^2$ equipped with the inner product

$$\langle \cdot , \cdot \rangle_{2,\alpha} = \langle \alpha D^2 \cdot , D^2 \cdot \rangle_0 \ .$$

Lemma 2.3. $\langle \hat{\mathscr{A}}_1 z, z\rangle \leq 0$ for every $z \in \mathrm{Dom}(\hat{\mathscr{A}}_1)$.

Proof:

$$\begin{aligned} \left\langle \hat{\mathscr{A}}_1 \binom{z_1}{z_2}, \binom{z_1}{z_2} \right\rangle &= \langle z_2, z_1 \rangle_{2,\alpha} + \langle -\alpha D^4 z_1, z_2 \rangle_0 \\ &= \int_0^1 \alpha D^2 z_1 D^2 z_2 + \int_0^1 -\alpha (D^4 z_1) z_2 \\ &= 0 \quad \text{for every} \quad z \in \mathrm{Dom}(\hat{\mathscr{A}}_1). \end{aligned}$$

Lemma 2.4. $R(\hat{\mathscr{A}}_1) = Z$.

Proof: We show that for every $(f,g) \in Z$, there exists a $(z_1, z_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_1)$ such that $\hat{\mathscr{A}}_1 \binom{z_1}{z_2} = \binom{f}{g}$: take $z_2 = f$ and

$z_1(x) = G_1(x) - xG_1(1) - (x^3/6 - x/6)G_1''(1)$ where $G_1(x) = -\frac{1}{\alpha}\int_0^x \int_0^{s_3} \int_0^{s_2}\int_0^{s_1} g(s)ds\,ds_1 ds_2 ds_3$. Then $(z_1,z_2) \in H_1^4 \times H_1^2 = \mathrm{Dom}(\hat{\mathscr{A}}_1)$ and $\hat{\mathscr{A}}_1\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ -\alpha D^4 z_1 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$.

Lemma 2.5. $\hat{\mathscr{A}}_1$ is densely defined. Hence H_1^2 is dense in H^0 and H_1^4 is dense in $H_1^2(\alpha)$.

Proof: Corollary 2.2 and $R(\hat{\mathscr{A}}_1) = Z$ and dissipativeness of $\mathscr{A}_1$ yield this result immediately.

Lemma 2.6. $\hat{\mathscr{A}}_1$ is skew-adjoint: $\hat{\mathscr{A}}_1^* = -\hat{\mathscr{A}}_1$.

Proof: First, we show $\hat{\mathscr{A}}_1^* \supset -\hat{\mathscr{A}}_1$. Let $z = (z_1,z_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_1)$, and find all w, v such that $\langle\hat{\mathscr{A}}_1 z,w\rangle = \langle z,v\rangle$. Thus $w,v \in Z$ must satisfy

$$\int_0^1 \alpha D^2 z_2 D^2 w_1 - \int_0^1 \alpha D^4 z_1 w_2 - \int_0^1 \alpha D^2 z_1 D^2 v_1 - \int_0^1 z_2 v_2 = 0,$$

where $w = (w_1,w_2)$, $v = (v_1,v_2)$. Or, integrating by parts,

$$\int_0^1 z_2(\alpha D^4 w_1 - v_2) + \alpha[(Dz_2)(D^2 w_1)]_0^1 - \int_0^1 \alpha(D^2 z_1)(D^2 w_2 + D^2 v_1) - \alpha[D^3 z_1 w_2]_0^1] = 0, \tag{2.6}$$

where we have applied the conditions on $z \in \mathrm{Dom}(\hat{\mathscr{A}}_1)$, $w \in Z$. Thus, if

$$\begin{aligned} v_2 &= \alpha D^4 w_1 \\ D^2 v_1 &= -D^2 w_2 \end{aligned} \qquad w \in H^4 \times H^2$$

and if $w_1''(0) = w_1''(1) = 0$ and $w_2(0) = w_2(1) = 0$, then clearly (2.6) clearly holds, or $-\hat{\mathscr{A}}_1$ is adjoint to $\hat{\mathscr{A}}_1$. We now show

$-\hat{\mathscr{A}}_1$ is the (maximal) adjoint operator; i.e., $\hat{\mathscr{A}}_1^* \subset -\hat{\mathscr{A}}_1$:

Let $g \in \mathrm{Dom}(\hat{\mathscr{A}}_1^*)$, $f = \hat{\mathscr{A}}_1^* g$; we show $f = -\hat{\mathscr{A}}_1 g$ and $g \in \mathrm{Dom}(\hat{\mathscr{A}}_1)$. We have

$$\langle z,f\rangle = \langle z,\hat{\mathscr{A}}_1^* g\rangle = \langle \hat{\mathscr{A}}_1 z,g\rangle \quad \text{for every} \quad z \in \mathrm{Dom}(\hat{\mathscr{A}}_1),$$

or, with $f = (f_1,f_2)$, $g = (g_1,g_2)$,

$$\int_0^1 \alpha(D^2 z_1)(D^2 f_1) + \int_0^1 z_2 f_2 - \alpha\int_0^1 (D^2 z_2)(D^2 g_1) + \int_0^1 \alpha(D^4 z_1) g_2 = 0,$$

for every $(z_1,z_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_1)$. Define $h_1(x) \equiv \int_0^x\int_0^{s_1} f_2(s)\,ds\,ds_1 - x\int_0^1\int_0^{s_1} f_2(s)\,ds\,ds_1$. Then $h_1(x)$ satisfies $h_1'' = f_2$ a.e., $h_1(0) = h_1(1) = 0$. Integrate the above by parts to obtain

$$\int_0^1 \alpha(D^4 z_1) f_1 + \alpha[(D^2 z_1) D f_1]_0^1 - \alpha[(D^3 z_1) f_1]_0^1 + \int_0^1 (D^2 z_2) h_1 + [z_2 (D h_1)]_0^1 - [(D z_2) h_1]_0^1 - \int_0^1 \alpha(D^2 z_2)(D^2 g_1) + \int_0^1 \alpha(D^4 z_1) g_2 = 0.$$

But, $f \in Z$ implies $f_1(0) = f_1(1) = 0$; also, $z \in \mathrm{Dom}(\hat{\mathscr{A}}_1)$ implies $z_2(0) = z_2(1) = (D^2 z_1)(0) = (D^2 z_1)(1) = z_1(0) = z_1(1) = 0$, and so this becomes

$$\alpha\int_0^1 (D^4 z_1)(f_1 + g_2) + \int_0^1 (D^2 z_2)(h_1 - \alpha D^2 g_1) = 0.$$

Since the first term is independent of the second, this is equivalent to the pair of equations

$$\int_0^1 (D^4 z_1)(f_1 + g_2) = 0$$

and

$$\int_0^1 (D^2 z_2)(h_1 - \alpha D^2 g_1) = 0.$$

Now, for every $v \in H_1^2$, there exists $z_1 \in H_1^4$ such that $D^4 z_1 = v$ so that $f_1 + g_2$ annihilates all $v \in H_1^2$ and so $f_1 + g_2 = 0$. So we have $g_2 = -f_1 \in H_1^2$.

Likewise, for every $w \in H^0$, there exists $z_2 \in H_1^2$ such that $D^2 z_2 = w$ and so $(h_1 - \alpha D^2 g_1)$ annihilates all $w \in H^0$. This implies $h_1 - \alpha D^2 g_1 = 0$. Hence, $D^2 g_1 = \frac{1}{\alpha} h_1 =$ $\frac{1}{\alpha}\left(\int_0^x\int_0^{s_1} f_2(s)ds ds_1 - x\int_0^1\int_0^{s_1} f_2(s)ds ds_1\right)$, $D^3 g_1 = \frac{1}{\alpha}\left(\int_0^x f_2(s)ds - \int_0^1\int_0^{s_1} f_2(s)ds ds_1\right)$, and $D^4 g_1 = \frac{1}{\alpha} f_2$ a.e., which implies $D^3 g_1$ is absolutely continuous and $D^4 g_1 \in H^0$ and so $g_1 \in H^4$. Furthermore, $(D^2 g_1)(0) = \frac{1}{\alpha} h_1(0) = 0$ and $(D^2 g_1)(1) = \frac{1}{\alpha} h_1(1) = 0$. This, along with the fact that $g \in Z = H_1^2 \times H^0$, implies $g_1 \in H_1^4$.

Thus, $g = (g_1, g_2) \in H_1^4 \times H_1^2 = \mathrm{dom}(\hat{\mathscr{A}}_1)$ and $(\hat{\mathscr{A}}_1^*)g = f = -\hat{\mathscr{A}}_1 g$, or $\hat{\mathscr{A}}_1^* \subset -\hat{\mathscr{A}}_1$. This proves the desired result, since $\hat{\mathscr{A}}_1^* \supset -\hat{\mathscr{A}}_1$. □

<u>Lemma 2.7</u>. $\hat{\mathscr{A}}_1$ is a closed, maximal dissipative operator, and is the generator of a C_0 semi-group of contractions on Z.

<u>Proof</u>: That $\hat{\mathscr{A}}_1$ is closed follows from

$$\hat{\mathscr{A}}_1^{**} = (\hat{\mathscr{A}}_1^*)^* = (-\hat{\mathscr{A}}_1)^* = -\hat{\mathscr{A}}_1^* = \hat{\mathscr{A}}_1.$$

We can also establish maximal dissipativeness easily by noting that

$$\langle \hat{\mathscr{A}}_1^* z, z\rangle = \langle -\hat{\mathscr{A}}_1 z, z\rangle = 0.$$

This implies that $\hat{\mathscr{A}}_1$ is maximal dissipative by a theorem of Krein [25, p. 87]. That $\hat{\mathscr{A}}_1$ is a generator then follows by another result [25, 4.5, p. 88].

Remark. We have shown that H_1^4 is dense in $H^2 \cap H_0^1$ in the $|\cdot|_{2,\alpha}$ norm by showing that $\hat{\mathscr{A}}_1$ is densely defined in Z. In this case the result can also be argued using the fact that Δ with $\mathrm{Dom}(\Delta) = H^2 \cap H_0^1$ is self-adjoint in H^0 and the results of Goldstein [18, p. 86].

Remark. Since $\hat{\mathscr{A}}_1$ is skew-adjoint, we could also have used Stone's theorem [18, p. 22] to argue that $\hat{\mathscr{A}}_1$ generates a C_0 group on Z. However we emphasize the role of dissipativeness since we require this in the case $\delta > 0$.

Case 2. This case corresponds to a beam clamped at both ends. Consider the operator in $Z = H_2^2(\alpha) \times H^0$ given by

$$\hat{\mathscr{A}}_2 = \begin{bmatrix} 0 & 1 \\ -\alpha D^4 & 0 \end{bmatrix} \quad \text{on} \quad \mathrm{Dom}(\hat{\mathscr{A}}_2) = H_2^4 \times H_2^2$$

where

$$H_2^4 = \{\phi \in H^4 \colon \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0\}$$

$$H_2^2 = \{\phi \in H^2 \colon \phi(0) = \phi'(0) = \phi(1) = \phi'(1) = 0\}$$

and

$$H_2^2(\alpha) = H_2^2 \text{ equipped with the inner product}$$

$$<\phi,\psi>_{2,\alpha} = <\alpha D^2\phi, D^2\psi>_0.$$

Lemma 2.8. $<\hat{\mathscr{A}}_2 z,z> \le 0$ for every $z \in \mathrm{Dom}(\hat{\mathscr{A}}_2)$.

Proof:

$$<\hat{\mathscr{A}}_2 z,z> = <z_2,z_1>_{2,\alpha} + <-\alpha D^4 z_1, z_2>_0$$

$$= \int_0^1 \alpha D^2 z_2 D^2 z_1 + \int_0^1 -\alpha D^4 z_1 z_2$$

$$= \int_0^1 \alpha D^2 z_2 D^2 z_1 - \alpha[(D^3 z_1) z_2]_0^1 + \alpha[(D^2 z_1)(D z_2)]_0^1 - \int_0^1 \alpha D^2 z_1 D^2 z_2$$

$= 0$ for every $z \in \mathrm{Dom}(\hat{\mathscr{A}}_2)$.

Lemma 2.9. $R(\hat{\mathscr{A}}_2) = Z$.

Proof: We show that for every $(f,g) \in Z$, there exists a $(z_1,z_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_2)$ such that $\hat{\mathscr{A}}_2\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$: take $z_2 = f$, and $z_1(x) = G_1(x) - (3x^2 - 2x^3)G_1(1) - (x^3-x^2)G_1'(1)$ where $G_1(x) = -\frac{1}{\alpha}\int_0^x\int_0^{s_3}\int_0^{s_2}\int_0^{s_1} g(s)\,ds\,ds_1\,ds_2\,ds_3$. Then, $(z_1,z_2) \in H_2^4 \times H_2^2 = \mathrm{Dom}(\hat{\mathscr{A}}_2)$ and $\hat{\mathscr{A}}_2\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ -\alpha D^4 z_1 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$.

Lemma 2.10. $\hat{\mathscr{A}}_2$ is skew-adjoint: $\hat{\mathscr{A}}_2^* = -\hat{\mathscr{A}}_2$.

Proof: First we show $\hat{\mathscr{A}}_2^* \supset -\hat{\mathscr{A}}_2$. Let $z = (z_1,z_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_2)$ and $w = (w_1,w_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_2)$. Then we show $\langle\hat{\mathscr{A}}_2 z,w\rangle = \langle z,-\hat{\mathscr{A}}_2 w\rangle$:

$$\langle\hat{\mathscr{A}}_2 z,w\rangle - \langle z,-\hat{\mathscr{A}}_2 w\rangle = \int_0^1 \alpha D^2 z_2 D^2 w_1 + \int_0^1 -\alpha(D^4 z_1)w_2 - \int_0^1 \alpha(D^2 z_1)(-D^2 w_2) - \int_0^1 z_2(\alpha D^4 w_1)$$

$$= 0.$$

Thus, $-\hat{\mathscr{A}}_2$ is adjoint to $\hat{\mathscr{A}}_2$. Now we must show $\hat{\mathscr{A}}_2^* \subset -\hat{\mathscr{A}}_2$.

Let $g \in \mathrm{Dom}(\hat{\mathscr{A}}_2^*)$, $f = \hat{\mathscr{A}}_2^* g$; we show $f = -\hat{\mathscr{A}}_2 g$. We have $\langle z,f\rangle = \langle z,\hat{\mathscr{A}}_2^* g\rangle = \langle\hat{\mathscr{A}}_2 z,g\rangle$ for every $z \in \mathrm{Dom}(\hat{\mathscr{A}}_2)$, or,

$$\int_0^1 \alpha D^2 z_1 D^2 f_1 + \int_0^1 z_2 f_2 - \alpha\int_0^1 D^2 z_2 D^2 g_1 + \alpha\int_0^1 D^4 z_1 g_2 = 0$$

for every $z = (z_1, z_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_2)$. Denoting $h_1(x) \equiv \int_0^x\int_0^{s_1} f_2(s)dsds + ax + b$, where a, b are arbitrary, $a \in R$, $b \in R$, and $h_2(s) \equiv \int_0^x f_2(s)ds + a$ we integrate by parts to obtain

$$\int_0^1 \alpha(D^4 z_1)f_1 + \alpha[(D^2 z_1)(Df_1)]_0^1 - \alpha[(D^3 z_1)f_1]_0^1 + \int_0^1 (D^2 z_2)h_1$$

$$+ [z_2 h_2]_0^1 - [(Dz_2)h_1]_0^1 - \alpha\int_0^1 (D^2 z_2)(D^2 g_1) + \int_0^1 \alpha(D^4 z_1)g_2 = 0.$$

But, $f \in Z$ implies $f_1 \in H_2^2$ and $f_2 \in H^0$, so the first two bracketed terms vanish; likewise $z \in \mathrm{Dom}(\hat{\mathscr{A}}_2)$ implies the last two terms in brackets vanish. Thus, we have

$$\int_0^1 \alpha(D^4 z_1)(f_1 + g_2) + \int_0^1 (D^2 z_2)(h_1 - \alpha D^2 g_1) = 0.$$

Since these two terms are independent, this is equivalent to the pair

$$\alpha\int_0^1 (D^4 z_1)(f_1 + g_2) = 0$$

and

$$\int_0^1 (D^2 z_2)(h_1 - \alpha D^2 g_1) = 0$$

for every $z \in \mathrm{Dom}(\hat{\mathscr{A}}_2) = H_2^4 \times H_2^4$. Thus $f_1 + g_2 = 0$ implies $g_2 = -f_1 \in H_2^2$;

$$D^2 g_1 = \frac{1}{\alpha} h_1 = \frac{1}{\alpha}\int_0^x\int_0^{s_1} f_2(s)ds + ax + b,\ D^3 g_1 = \frac{1}{\alpha}\int_0^x f_2(s)ds + a$$

is absolutely continuous, $D^4 g_1 = \frac{1}{\alpha} f_2 \in H^0$ implies $g_1 \in H^4$ and $g \in Z$ implies $g_1 \in H_2^2$, so that $g_1 \in H^4 \cap H_2^2 = H_2^4$. Hence

$g \in \mathrm{Dom}(\hat{\mathscr{A}}_2)$ with $\hat{\mathscr{A}}_2^* g = f = -\hat{\mathscr{A}}_2 g$, or $\hat{\mathscr{A}}_2^* \subset -\hat{\mathscr{A}}_2$. Thus $\hat{\mathscr{A}}_2^* = \hat{\mathscr{A}}_2$.

Lemma 2.11. $\hat{\mathscr{A}}_2$ is a closed, maximal dissipative operator and is the generator of a C_0 semi-group of contractions on Z.

Proof: Identical to the proof of Lemma 6.

Remark. The statement H_2^4 is dense in H_2^2 (in the usual H^2 norm) follows directly from the fact that $C_0^\infty = \{\phi \in C^\infty : \phi$ has compact support in $(0,1)\}$ is dense in H_2^2 and $C_0^\infty \subset H_2^4 \subset H_2^2$ (cf: [30, p. 91]).

Case 3. Finally consider the case corresponding to a cantilevered beam. Consider the operator in $H_3^2(\alpha) \times H^0$ given by

$$\hat{\mathscr{A}}_3 = \begin{pmatrix} 0 & 1 \\ -\alpha D^4 & 0 \end{pmatrix} \quad \text{on} \quad \mathrm{Dom}(\hat{\mathscr{A}}_3) = H_3^4 \times H_3^2$$

where

$$H_3^4 = \{\phi \in H^4 : \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0\}$$

$$H_3^2 = \{\phi \in H^2 : \phi(0) = \phi'(0) = 0\}$$

and $H_3^2(\alpha)$ is H_3^2 equipped with the inner product $\langle \cdot , \cdot \rangle_{2,\alpha} = \langle \alpha D^2 \cdot , D^2 \cdot \rangle_0$.

Lemma 2.12. $\langle \hat{\mathscr{A}}_3 z, z\rangle \leq 0$ for every $z \in \mathrm{Dom}(\hat{\mathscr{A}}_3)$.

Proof:

$$\begin{aligned}\langle \hat{\mathscr{A}}_3 z, z\rangle &= \langle z_2, z_1\rangle_{2,\alpha} + \langle -\alpha D^4 z_1, z_2\rangle_0 \\ &= \int_0^1 \alpha (D^2 z_2)(D^2 z_1) + \int_0^1 -\alpha (D^4 z_1) z_2\end{aligned}$$

$g \in \mathrm{Dom}(\hat{\mathscr{A}}_2)$ with $\hat{\mathscr{A}}_2^* g = f = -\hat{\mathscr{A}}_2 g$, or $\hat{\mathscr{A}}_2^* \subset -\hat{\mathscr{A}}_2$. Thus $\hat{\mathscr{A}}_2^* = \hat{\mathscr{A}}_2$.

Lemma 2.11. $\hat{\mathscr{A}}_2$ is a closed, maximal dissipative operator and is the generator of a C_0 semi-group of contractions on Z.

Proof: Identical to the proof of Lemma 6.

Remark. The statement H_2^4 is dense in H_2^2 (in the usual H^2 norm) follows directly from the fact that $C_0^\infty = \{\phi \in C^\infty$: ϕ has compact support in $(0,1)\}$ is dense in H_2^2 and $C_0^\infty \subset H_2^4 \subset H_2^2$ (cf: [30, p. 91]).

Case 3. Finally consider the case corresponding to a cantilevered beam. Consider the operator in $H_3^2(\alpha) \times H^0$ given by

$$\hat{\mathscr{A}}_3 = \begin{pmatrix} 0 & 1 \\ -\alpha D^4 & 0 \end{pmatrix} \quad \text{on} \quad \mathrm{Dom}(\hat{\mathscr{A}}_3) = H_3^4 \times H_3^2$$

where

$$H_3^4 = \{\phi \in H^4 \colon \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0\}$$

$$H_3^2 = \{\phi \in H^2 \colon \phi(0) = \phi'(0) = 0\}$$

and $H_3^2(\alpha)$ is H_3^2 equipped with the inner product $\langle\cdot,\cdot\rangle_{2,\alpha} = \langle\alpha D^2\cdot, D^2\cdot\rangle_0$.

Lemma 2.12. $\langle\hat{\mathscr{A}}_3 z, z\rangle \leq 0$ for every $z \in \mathrm{Dom}(\hat{\mathscr{A}}_3)$.

Proof:

$$\langle\hat{\mathscr{A}}_3 z, z\rangle = \langle z_2, z_1\rangle_{2,\alpha} + \langle -\alpha D^4 z_1, z_2\rangle_0$$

$$= \int_0^1 \alpha(D^2 z_2)(D^2 z_1) + \int_0^1 -\alpha(D^4 z_1) z_2$$

$$= \int_0^1 \alpha(D^2z_2)(D^2z_1) - \alpha[(D^3z_1)z_2]_0^1 + \alpha[(D^2z_1)(Dz_2)]_0^1$$
$$- \int_0^1 \alpha(D^2z_1)(D^2z_2)$$
$$= 0 \quad \text{for every} \quad z \in \mathrm{Dom}(\hat{\mathscr{A}}_3).$$

<u>Lemma 2.13</u>. $R(\hat{\mathscr{A}}_3) = Z$.

<u>Proof</u>: We show that for every $(f,g) \in Z$, there exists a $(z_1,z_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_3)$ such that $\hat{\mathscr{A}}_3\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$: take $z_2 = f$, and $z_1(x) = G_1(x) - \frac{x^2}{2} G_1''(1) - (\frac{x^3}{6} - \frac{x^2}{2})G_1'''(1)$ where $G_1(x) = \int_0^x\int_0^{s_3}\int_0^{s_2}\int_0^{s_1} g(s)ds ds_1 ds_2 ds_3$. Then, $(z_1,z_2) \in H_3^4 \times H_3^2 = \mathrm{Dom}(\hat{\mathscr{A}}_3)$ and $\hat{\mathscr{A}}_3\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ -\alpha D^4 z_1 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}$.

<u>Lemma 2.14</u>. $\hat{\mathscr{A}}_3$ is skew-adjoint: $\hat{\mathscr{A}}_3^* = -\hat{\mathscr{A}}_3$.

<u>*Proof:*</u> *First we show* $\hat{\mathscr{A}}_3^* \supset -\hat{\mathscr{A}}_3$. *Let* $z = (z_1,z_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_3)$ and $w = (w_1,w_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_3)$. Then we show $\langle\hat{\mathscr{A}}_3 z,w\rangle = \langle z,-\hat{\mathscr{A}}_3 w\rangle$:

$$\langle\hat{\mathscr{A}}_3 z,w\rangle - \langle z,-\hat{\mathscr{A}}_3 w\rangle = \int_0^1 \alpha D^2z_2D^2w_1 + \int_0^1 -\alpha(D^4z_1)w_2$$
$$- \int_0^1 \alpha(D^2z_1)(-D^2w_2) - \int_0^1 z_2(\alpha D^4w_1)$$
$$= 0.$$

Thus, $-\hat{\mathscr{A}}_3$ is adjoint to $\hat{\mathscr{A}}_3$. Now we must show $\hat{\mathscr{A}}_3^* \subset -\hat{\mathscr{A}}_3$.

Let $g \in \mathrm{Dom}(\hat{\mathscr{A}}_3^*)$, $f = \hat{\mathscr{A}}_3^* g$; we show $f = -\hat{\mathscr{A}}_3 g$. We have $\langle z,f\rangle = \langle z,\hat{\mathscr{A}}_3^* g\rangle = \langle\hat{\mathscr{A}}_3 z,g\rangle$ for every $z \in \mathrm{Dom}(\hat{\mathscr{A}}_3)$, or,

$$\int_0^1 \alpha(D^2z_1)(D^2f_1) + \int_0^1 z_2f_2 - \alpha\int_0^1(D^2z_2)(D^2g_1) + \alpha\int_0^1(D^4z_1)g_2 = 0$$

for every $z = (z_1,z_2) \in \mathrm{Dom}(\hat{\mathscr{A}}_3)$. Denote by $h(x)$ the function

$$h(x) \equiv \int_0^x \int_0^{s_1} f_2(s)dsds_1 - x\int_0^1 f_2(d)ds - \int_0^1 \int_0^{s_1} f_2(s)dsds_1 + \int_0^1 f_2(s)ds$$

satisfying $D^2h = f_2$, $h(1) = h'(1) = 0$. Integrate by parts to obtain

$$\int_0^1 \alpha(D^4z_1)f_1 + \alpha[(D^2z_1)(Df_1)]_0^1 - \alpha[(D^3z_1)f_1]_0^1 + \int_0^1 (D^2z_2)h$$
$$+ [z_2(Dh)]_0^1 - [(Dz_2)h] - \alpha\int_0^1 (D^2z_2)(D^2g_1) + \alpha\int_0^1 (D^4z_1)g_2 = 0.$$

But $f = (f_1,f_2) \in Z$ implies $f_1(0) = f_1'(0)$; likewise, $z \in \mathrm{Dom}(\hat{\mathscr{A}}_3)$ implies that $z_1(0) = z_1'(0) = z_1''(1) = z_1'''(1) = 0$ and $z_2'(0) = z_2(0) = 0$. Thus the above becomes

$$\int_0^1 \alpha(D^4z_1)(f_1+g_2) + \int_0^1 (D^2z_2)(h - \alpha D^2g_1) = 0.$$

Since the first term is independent of the second, this is equivalent to the pair

$$\int_0^1 (D^4z_1)(f_1 + g_2) = 0$$

and

$$\int_0^1 (D^2z_2)(h - \alpha D^2g_1) = 0.$$

Thus, $f_1 + g_2 = 0$ which implies $g_2 = -f_1 \in H_3^2$. Furthermore, $h - \alpha D^2g_1 = 0$ implies $D^2g_1 = \frac{1}{\alpha} h$, $D^3g_1 = \frac{1}{\alpha}\left(\int_0^x f_2(s)ds - \int_0^1 f_2(s)ds\right)$, and $D^4g_1 = \frac{1}{\alpha} f_2$, so that D^3g_1 is absolutely continuous and $D^4g_1 \in H^0$ which implies $g_1 \in H^4$. Also $(D^2g_1)(1) = \frac{1}{\alpha} h(1) = 0$ and $(D^3g_1)(1) = \frac{1}{\alpha} h'(1) = 0$. This, together with the fact that $g = (g_1,g_2) \in Z = H_3^2 \times H^0$, implies that $g \in H_3^4 \times H_3^2 = \mathrm{Dom}(\hat{\mathscr{A}}_3)$, with $(\hat{\mathscr{A}}_3^*)g = f = -\hat{\mathscr{A}}_3 g$, or $\hat{\mathscr{A}}_3^* \subset -\hat{\mathscr{A}}_3$. This proves the desired result, since we already have $\hat{\mathscr{A}}_3^* \supset -\hat{\mathscr{A}}_3$. □

Lemma 2.15. $\hat{\mathscr{A}}_3$ is a closed, maximal dissipative operator and is the generator of a C_0 semigroup of contractions on Z.

Proof: Identical to the proof of Lemma 6.

Case 4. Now consider the general case corresponding to a simply supported beam with internal damping. Define the operator $\mathscr{A}_1$ in $H_1^2(\alpha) \times H^0 = Z$ by

$$\mathscr{A}_1 = \begin{bmatrix} 0 & 1 \\ -\alpha D^4 & -\delta D^4 \end{bmatrix} \text{ on } \mathrm{Dom}(\mathscr{A}_1) = H_1^4 \times H_1^4$$

where $\alpha \in R$, $\alpha > 0$, $\delta \in R$, $\delta > 0$,

$$H_1^2 = \{\phi \in H^2 : \phi(0) = \phi(1) = 0\}$$

$$H_1^4 = \{\phi \in H^4 : \phi(0) = \phi''(0) = \phi(1) = \phi''(1) = 0\}$$

and $H_1^2(\alpha)$ is H_1^2 with the inner product $\langle\cdot,\cdot\rangle_{2,\alpha} = \langle\alpha D^2\cdot, D^2\cdot\rangle_0$. By the results of Section 1.2, H_1^4 is dense in H^0 and by Lemma 2.5 above, H_1^4 is dense in $H_1^2(\alpha)$; thus $\mathscr{A}_1$ is densely defined.

Lemma 2.16. $\mathscr{A}_1$ is dissipative.

Proof: Let $z = (z_1, z_2) \in \mathrm{Dom}(\mathscr{A}_1)$. Then,

$$\begin{aligned} \langle\mathscr{A}z, z\rangle &= \langle z_2, z_1\rangle_{2,\alpha} + \langle -\alpha D^4 z_1, z_2\rangle_0 + \langle -\delta D^4 z_2, z_2\rangle_0 \\ &= \alpha\int_0^1 (D^2 z_2)(D^2 z_1) + \int_0^1 (-\alpha D^4 z_1) z_2 + \int_0^1 (-\delta D^4 z_2) z_2 \\ &= -\alpha([(D^3 z_1) z_2]_0^1 - [(D^2 z_1)(D z_2)]_0^1) - \delta\int_0^1 (D^2 z_2)^2 \\ &\quad - \delta([(D^3 z_2) z_2]_0^1 - [(D^2 z_2)(D z_2)]_0^1) = -\delta\int_0^1 (D^2 z_2)^2 \\ &\leq 0 \text{ for } \delta \geq 0. \quad \square \end{aligned}$$

Remark. Note that we required $(D^2 z_2)(0) = (D^2 z_2)(1) = 0$ to establish the dissipative inequality, and this is one reason for including these conditions in $\mathrm{Dom}(\mathscr{A}_1)$.

$\mathscr{A}_1$ itself does not generate a C_0 semigroup on Z, since $R(\mathscr{A}_1 - \lambda I) \neq Z$. However, $\mathscr{A}_1$ does possess a maximal dissipative extension $\tilde{\mathscr{A}}_1$ which is closed (cf: [25, p. 87]). In the case $\delta = 0$, we have seen that the maximal dissipative extension is $\hat{\mathscr{A}}_1$. Since $\mathrm{Dom}(\mathscr{A}_1)$ is dense in Z, the following dense inclusions hold:

$$\mathrm{Dom}(\mathscr{A}_1) \subset \mathrm{Dom}(\tilde{\mathscr{A}}_1) \subset Z,$$

and the maximal dissipativeness of $\tilde{\mathscr{A}}_1$ is sufficient to ensure that it is the generator of a C_0 semigroup of contractions on Z.

Case 5. Consider next the operator in $H_2^2(\alpha) \times H^0$ given by

$$\mathscr{A}_2 = \begin{bmatrix} 0 & 1 \\ -\alpha D^4 & -\delta D^4 \end{bmatrix} \quad \text{on} \quad \mathrm{Dom}(\mathscr{A}_2) = H_2^4 \times H_2^4, \ \alpha, \delta > 0.$$

This corresponds to the beam clamped at both ends as in Case 2, but with internal damping included.

Lemma 2.17. $\mathscr{A}_2$ is dissipative.

Proof: See Lemma 2.16. The same integration by parts yields

$$\langle \mathscr{A}_2 z, z \rangle = -\delta \int_0^1 (D^2 z_2)^2 \leq 0$$

for every $z = (z_1, z_2) \in \mathrm{Dom}(\mathscr{A}_2)$. □

Since $R(\mathscr{A}_2 - \lambda I) \neq Z$, we will again need to take a maximal dissipative extension $\tilde{\mathscr{A}}_2$ where

$$\mathrm{Dom}(\mathscr{A}_2) \subset \mathrm{Dom}(\tilde{\mathscr{A}}_2) \subset Z$$

are dense inclusions. Lemma 2.18 and a theorem of Krein [25, 4.3, p. 87] are sufficient to guarantee the existence of $\tilde{\mathscr{A}}_2$. In the case $\delta = 0$, we have seen that $\tilde{\mathscr{A}}_2 = \hat{\mathscr{A}}_2$. By another standard result ([25, 4.4, p. 87]) $\tilde{\mathscr{A}}_2$ is closed, and generates a C_0 semigroup of contractions on Z.

Case 6. For the final case we wish to consider, we take the operator in $Z = H_3^2(\alpha) \times H^0$ defined by

$$\mathscr{A}_3 = \begin{pmatrix} 0 & 1 \\ -\alpha D^4 & -\delta D^4 \end{pmatrix} \quad \text{on} \quad \mathrm{Dom}(\mathscr{A}_3) = H_3^4 \times H_3^4$$

where

$$H_3^4 = \{\phi \in H^4\colon \phi(0) = \phi'(0) = \phi''(1) = \phi'''(1) = 0\},$$

$$H_3^2 = \{\phi \in H^2\colon \phi(0) = \phi(1) = 0\},$$

and

$$H_3^2(\alpha) = H_3^2 \text{ equipped with the inner product}$$

$$\langle \cdot,\cdot \rangle_{2,\alpha} = \langle \alpha D^2\cdot, D^2\cdot \rangle_0 .$$

Lemma 2.18. $\mathscr{A}_3$ is dissipative.

Proof: See Lemma 2.16. The same integration by parts yields

$$\langle \mathscr{A}_3 z, z \rangle = -\delta \int_0^1 (D^2 z_2)^2 \le 0 \quad \text{for every} \quad z = (z_1, z_2) \in \mathrm{Dom}(\mathscr{A}_3).$$

$\mathscr{A}_3$ has a maximal dissipative extension $\tilde{\mathscr{A}}_3$, by the theorem of Krein [25, 4.3, p. 87], and since $\mathrm{Dom}(\mathscr{A}_3)$ is dense in Z, $\mathrm{Dom}(\mathscr{A}_3) \subset \mathrm{Dom}(\tilde{\mathscr{A}}_3) \subset Z$ where all inclusions are dense. Thus [25, p. 88] $\tilde{\mathscr{A}}_3$ generates a C_0 semigroup of contractions on Z.

We shall require certain dense subsets of smooth functions in $\mathrm{Dom}(\tilde{\mathscr{A}}_k)$ to apply Proposition 1.5 (Trotter-Kato theorem). To construct these dense subsets, we will use eigenfunctions of D^4 on $\mathrm{Dom}(D^4) = H_k^4$.

We first note that if $u \in H_k^2$, $k = 1,2$, or 3, then

$$\max |u(x)| \leq K|u|_2,$$

and

$$|u(x_1) - u(x_2)| \leq K\sqrt{x_1-x_2}\ |u|_2.$$

These inequalities follow from an application of the Cauchy-Schwartz inequality and from the fact that for $u \in H_k^2$, $|u|_1 \leq K|u|_2$ for some constant K (this is true because $|\cdot|_1$ and $|\cdot|_2$ are equivalent to the usual H^1 and H^2 norms respectively on H_k^2 and $|u|_{H^1} \leq |u|_{H^2}$). In particular, for $u \in H_k^2$, $x_1, x_2 \in [0,1]$, $u(x_2) = u(x_1) + \int_{x_1}^{x_2} u'(\xi)d\xi$ implies

$$|u(x_2)-u(x_1)|^2 \leq \left|\int_{x_1}^{x_2} u'(\xi)d\xi\right|^2$$

$$\leq |x_2-x_1|\ |u'|_0^2, \text{ by the Schwartz inequality,}$$

$$= |x_2-x_1|\ |u|_1^2 \leq K^2|x_2-x_1|\ |u|_2^2.$$

Then we obtain from this (with $x_1 = 0$) that

$$|u(x_2)| \leq K\sqrt{x_2}\ |u|_2 \quad \text{for every } x_2 \in [0,1]$$

which implies the first inequality. Thus an application of the Ascoli Theorem (see [38, pp. 249-250]) implies that from every sequence $\{u_n(x)\}$ which is bounded in the norm of H_k^2, it is possible to select a uniformly convergent subsequence. Additionally, for any two elements $u,v \in H_k^2$,

$$|\langle u,v\rangle_0| \le \max|u|\ \max|v| \le K^2|u|_2|v|_2.$$

Now let B_k, $k = 1,2$, or 3 be the operator D^4 in H_k^2 with $\mathrm{Dom}(B_k) = H_k^4$. B_k^{-1} exists (see proofs of Lemmas 2.4, 2.9, and 2.13), and

$$\langle B_k^{-1}u,v\rangle_2 = \langle u,v\rangle_0 \quad \text{for every } u,v \in H_k^2.$$

B_k^{-1} is clearly symmetric and furthermore it is a compact operator on H_k^2: given a bounded sequence $\{u_n(x)\}$ in H_k^2, we can select a uniformly convergent subsequence $\{v_n(x)\}$, and

$$\begin{aligned}\langle B_k^{-1}(v_n-v_m),v_n-v_m\rangle_2 &= \langle v_n-v_m,v_n-v_m\rangle_0\\ &= \int_0^1 |v_n-v_m|^2dx \longrightarrow 0,\end{aligned}$$

and thus [38, p. 206] B_k^{-1} is a compact operator from $X = H_k^2$ into X.

For the moment, fix $k = 1,2$, or 3. Since B_k^{-1} is symmetric and compact, standard results [45, p. 343] imply that B_k^{-1} has a complete orthonormal set of eigenfunctions ψ_n and associated eigenvalues λ_n, and $f = \sum_1^\infty \langle f,\psi_n\rangle_2\psi_n$ for every $f \in \mathrm{Dom}(B_k) = H_k^4$. We have $B_k\psi_n = \mu_n\psi_n$, where $\mu_n = 1/\lambda_n$.

We can construct a dense set in H^0 by taking $\phi_n = \sqrt{\mu_n}\ \psi_n$. Then, since for every $f \in H_k^4$,

$$\langle f,\psi_n\rangle_2 = \langle D^2f,D^2\psi_n\rangle_0 = \langle f,D^4\psi_n\rangle_0 = \langle f,\mu_n\psi_n\rangle_0 = \langle f,\phi_n\rangle_0\ \sqrt{\mu_n},$$

we have $|\sum_1^m \langle f,\psi_n\rangle_2\psi_n - f|_2 \longrightarrow 0$ implies that $|\sum_1^m \langle f,\phi_n\rangle_0\phi_n - f|_2 \longrightarrow 0$, which implies, since the norm $|\cdot|_2$ is stronger than the $|\cdot|_0$-norm, that $|\sum_1^m \langle f,\phi_n\rangle\phi_n - f|_0 \longrightarrow 0$ for every $f \in H_k^4$. As a consequence, the set $\mathrm{span}(\{\phi_n\})$ is dense in H^0, since H_k^4 is $|\cdot|_0$-dense in H^0. Also, $B_k\phi_n = \mu_k\phi_n$ for $n = 1,2,\ldots$.

The smoothness of the functions ψ_n, ϕ_n (we shall only require H^6 smoothness) follows from the fact that $B_k^2\psi_n = \mu_k^2\psi_n$ implies that $\psi_n \in \mathrm{Dom}(B_k^2) \subset H^8$, and similarily for ϕ_n.

Thus for our dense subsets of smooth functions in $Z = H_k^2 \times H^0$, we are led to define

$$\gamma_{1,j} = \begin{pmatrix} \psi_j \\ 0 \end{pmatrix}, \gamma_{2,j} = \begin{pmatrix} 0 \\ \phi_j \end{pmatrix}.$$

Let $\mathscr{D}^N = \mathrm{span}\{\gamma_{1,j}\}_{j=1}^N \cup \mathrm{span}\{\gamma_{2,j}\}_{j=1}^N$, and $\mathscr{D} = \bigcup_1^\infty \mathscr{D}^N$. Clearly, by the above, $\mathscr{D}$ is a dense subset Z, and $\mathscr{D} \subset \mathrm{Dom}(\mathscr{A}_k) \subset \mathrm{Dom}(\tilde{\mathscr{A}}_k) \subset Z$.

Remark. The set $\{\phi_j\}$ associated with B_k^{-1} and the set $\mathscr{D}$ constructed from the ϕ_j will clearly be a different set of functions for each k. Since their properties are the same, we denote these by the generic symbols $\{\phi_j\}$ and $\mathscr{D}$ where it is understood by context that these represent the functions in H_k^2 and $H_k^2 \times H^0$ respectively.

We require one further property of $\mathscr{D}$ to apply Proposition 1.5 (Trotter-Kato). We must show

(2.7) $$\overline{(\tilde{\mathscr{A}}_k - \lambda_0 I)\mathscr{D}} = Z \quad \text{for some } \lambda_0 > 0.$$

First observe that $\tilde{\mathscr{A}}_k - \lambda_0 I$ is invertible for $\lambda_0 > 0$, since $\tilde{\mathscr{A}}_k$ generates a contraction semigroup on Z, and further that $\tilde{\mathscr{A}}_k = \mathscr{A}_k$ when restricted to $\mathrm{Dom}(\mathscr{A}_k)$. Then since $\mathscr{D} \subset \mathrm{Dom}(\mathscr{A}_k)$,

$$(\tilde{\mathscr{A}}_k - \lambda_o I)\mathscr{D} = (\mathscr{A}_k - \lambda_0 I)\mathscr{D} .$$

In fact, we show that $(\tilde{\mathscr{A}}_k - \lambda_0 I)\mathscr{D} = \mathscr{D}$ and the result (2.7) follows from the denseness of $\mathscr{D}$ in Z.

<u>Lemma 2.19.</u> $(\tilde{\mathscr{A}}_k - \lambda_0 I)\mathscr{D} = \mathscr{D}$ for some $\lambda_0 > 0$.

<u>Proof</u>: a) $(\tilde{\mathscr{A}}_k - \lambda_0 I)\mathscr{D} \subset \mathscr{D}$ is trivial.

b) $(\tilde{\mathscr{A}}_k - \lambda_0 I)\mathscr{D} \supset \mathscr{D}$: Since $(\tilde{\mathscr{A}}_k - \lambda_0 I)^{-1}$ exists, it suffices to show that for a typical basis element this holds. We do the typical basis elements $\gamma_{1,j}$ and $\gamma_{2,j}$ separately. We show that

i) there exists a $z = (z_1, z_2) \in \mathscr{D}$ such that

$$(\tilde{\mathscr{A}}_k - \lambda_0 I)z - \gamma_{2,j} = (\phi_j).$$

This holds iff $(\mathscr{A}_1 - \lambda_0 I)z = \gamma_{2,j}$

$$\text{iff} \quad \begin{cases} -\lambda_0 z_1 + z_2 = 0 \\ -\alpha D^4 z_1 - \delta D^4 z_2 - \lambda_0 z_2 = \phi_j \end{cases}$$

$$\text{iff} \quad -\alpha D^4 z_1 - \delta\lambda_0 D^4 z_1 - \lambda_0^2 z_1 = \phi_j$$

$$\text{iff} \quad -(\alpha + \delta\lambda_0) D^4 z_1 - \lambda_0^2 z_1 = \phi_j ;$$

with $z_1 = a\phi_j$, since $D^4\phi_j = \mu_j\phi_j$, $j \geq 1$, this can be solved iff $a(-(\alpha + \delta\lambda_0)\mu_j - \lambda_0^2) = 1$, or,

$$a = \frac{1}{\lambda_0^2 + (\alpha + \delta\lambda_0)\mu_j}$$

Since all quantities are positive, this can always be solved. For the other typical basis element $\gamma_{1,j} = \binom{\psi_j}{0}$, we must show that

ii) there exists a $z = (z_1, z_2) \in \mathscr{D}$ such that

$(\tilde{\mathscr{A}}_1 - \lambda_0 I)z = \gamma_{2,j}$, which holds

iff $(\mathscr{A}_1 - \lambda_0 I)z = \gamma_{2,j}$

$$\text{iff} \begin{cases} -\lambda_0 z_1 + z_2 = \psi_j \\ -\alpha D^4 z_1 - \delta D^4 z_2 - \lambda_0 z_2 = 0; \end{cases}$$

$$\text{iff} \quad -(\alpha + \delta\lambda_0)D^4 z_1 - \lambda_0^2 z_1 = \delta D^4 \psi_j + \lambda_0 \psi_j;$$

with $z_1 = a\psi_j$, this can be solved for $(z_1, z_2) \in \mathscr{D}$ (using $D^4\psi_j = \mu_j \psi_j$)

$$\text{iff} \quad -(\alpha + \delta\lambda_0)\mu_j a - \lambda_0^2 a = \delta\mu_j + \lambda_0,$$

or

$$a = -\frac{\delta\mu_j + \lambda_0}{(\alpha + \delta\lambda_0)\mu_j + \lambda_0^2}.$$

Since all quantities in the denominator are positive, this solution is possible (a finite) for any $\lambda_0 > 0$. □

Thus for each $R = 1,2,$ or 3 we have a set $\mathscr{D}$ such that

$$\mathscr{D} \subset \text{Dom}(\mathscr{A}_k) \subset \text{Dom}(\tilde{\mathscr{A}}_k) \subset Z,$$

where all inclusions are dense, and

$$\overline{(\tilde{\mathscr{A}}_k - \lambda_0 I)\mathscr{D}} = \overline{(\mathscr{A}_k - \lambda_0 I)\mathscr{D}} = Z$$

for any $\lambda_0 > 0$.

We can summarize the previous results. The operators $\mathscr{A}_k$, $k = 1,2,3$ defined in $H_k^2(\alpha) \times H^0$ with $\text{Dom}(\mathscr{A}_k) = H_k^4 \times H_k^4$ have a maximal dissipative extension $\tilde{\mathscr{A}}_k$ on $\text{Dom}(\tilde{\mathscr{A}}_k)$ where $\text{Dom}(\mathscr{A}_k) \subset \text{Dom}(\tilde{\mathscr{A}}_k) \subset Z$ densely. Furthermore, for each operator

$\tilde{\mathscr{A}}_k$, k = 1,2,3, there exists a set $\mathscr{D}$ consisting of smooth functions such that $\bar{\mathscr{D}} = Z$ and $\overline{(\tilde{\mathscr{A}}_k - \lambda I)\mathscr{D}} = Z$.

We now consider the approximate problem. Define for each k, k = 1,2,3, the spaces Z and Z^N by

$$Z^N = S_k^5(\Delta^N) \times S_k^5(\Delta^N), \quad Z = Z(q) = Z(\alpha) = H_k^2(\alpha) \times H^0$$

(recall $\alpha = q_1$), and let $P^N(q)$ be the orthogonal projection $P^N(q)\colon Z \longrightarrow Z^N$. The approximate problem corresponds to solutions of

$$\dot{z}^N(t) = \mathscr{A}^N(q)z^N(t) + F^N(q,t)$$
$$z^N(0) = z_0^N(q)$$

where $\mathscr{A}^N(q) = P^N(q)\mathscr{A}(q)P^N(q)$, $F^N(q,t) = P^N(q)F(q,t)$, and $z_0^N(q) = P^N(q)z_0(q)$.

We give a concrete realization of these approximating equations at the end of this section, along with numerical results in Section 4. Our first goal is to prove convergence of $\bar{q}^N \longrightarrow \bar{q}$ in the context of Chapter 1. That is, we prove that solutions of the approximate identification problem (IDN) converge to solutions of the full identification problem (IDA), using Proposition 1.4.

We first prove the approximating subspaces converge in the appropriate sense.

Lemma 2.20. Assuming (HQ), the projections $P^N(q)$ converge strongly to the identity I in Z as $N \to \infty$, for $k = 1$ or 2.

Proof: Let $z = (z_1, z_2) \in \mathscr{D}$ where $\mathscr{D}$ is the dense subset in Z defined in Lemma 2.19 for $k = 1,2$ respectively. Then $P^N(q)z \equiv (z_1^N, z_2^N) \equiv (P_3^N z_1, P_2^N z_2)$, where P_3^N is the projection of the first coordinate onto $S_k^5(\Delta^N)$ in the $|\cdot|_{2,\alpha}$ norm and P_2^N is the projection of the second coordinate in the $|\cdot|_0$ norm. Then,

$$|P^N(q)z-z|^2 = |P_3^N z_1 - z_1|^2_{2,} + |P_2^N z_2 - z_2|^2_0$$
$$\leq (\sqrt{\alpha}\ \kappa_{3,0}\ (\tfrac{1}{N})^4 |D^6 z|_0)^2 + (\sqrt{\alpha}\ \kappa_{2,0}(\tfrac{1}{N})^6 |D^6 z|_0),$$

where we have applied Lemmas 1.18 and 1.19, which implies $|P^N(q)z-z| \to 0$ as $N \to \infty$ for $z \in \mathscr{D}$. But since $\mathscr{D}$ is dense in Z and the projections $P^N(q)$ are bounded operators, it follows that $|P^N(q)z-z| \longrightarrow 0$ for any $z \in Z$. □

We may now state and prove the main theorem of this section.

Theorem 2.21. Let (H) hold. Then the semigroups $T(t;q)$ and $T^N(t;q)$ generated by $\tilde{\mathscr{A}}_k(q)$ and $\mathscr{A}_k^N(q)$ respectively, $k = 1$ or 2, satisfy $\|T(t;q)\| \leq 1$ and $\|T^N(t;q)\| \leq 1$. Moreover, for any sequence $\{q^N\}$ converging to q^* in Q, we have

$$|T^N(t;q^N)z - T(t;q^*)z| \longrightarrow 0 \text{ uniformly on } [0,T]$$

for each $z \in Z$.

Proof: The bound $\|T(t;q)\| \leq 1$ follows because $\tilde{\mathscr{A}}_k$ generates a C_0 semigroup of contractions on Z. Also by the remarks in Chapter 1, Section 1, it follows that $\|T^N(t;q)\| \leq 1$. To establish the convergence results, we apply the Trotter-Kato theorem (Proposition 1.5). Let $\mathscr{B} = Z(q^*), \mathscr{B}^N = Z^N(q^N)$; also take $\mathscr{A} = \tilde{\mathscr{A}}_k(q^*)$ and $\mathscr{A}^N = \mathscr{A}_k^N(q^N)$. Let $\pi^N: Z(q^*) \longrightarrow Z(q^N)$ be the cononical isomorphism between $Z(q^*)$ and $Z(q^N)$. Then $q^N \longrightarrow q^*$ implies $|\pi^N z| \longrightarrow |z|$, verifying hypothesis i) of Trotter-Kato. Thus it remains to verify iii). We have already defined for each k a set $\mathscr{D}$ such that $(\tilde{\mathscr{A}}_k(q^*) - \lambda I)\mathscr{D}$ is dense in $\mathscr{B}$. To establish convergence, we see that for each $z = (z_1, z_2) \in \mathscr{D}$ (suppressing the notation π^N)

$$(2.8)\quad |\mathscr{A}^N(q^N)z - \mathscr{A}(q^*)z| = |P^N(q^N)\mathscr{A}(q^N)P^N(q^N) - \mathscr{A}(q^*)z|$$
$$\leq |(\mathscr{A}(q^N) - \mathscr{A}(q^*))P^N(q^N)z|$$
$$+ |\mathscr{A}(q^*)(P^N(q^N)z - z)| + |(P^N(q^N) - I)\mathscr{A}(q^*)z|.$$

We bound each of these terms separately.

The second term can be written explicitly as

$$\mathscr{A}(q^*)(P^N(q^N)z-z) = \begin{pmatrix} 0 & 1 \\ -q_1^*D^4 & -q_2^*D^4 \end{pmatrix}(P^N(q^N)z-z)$$
$$= \begin{pmatrix} z_2^N - z_2 \\ -q_1^*D^4(z_1^N - z_1) - q_2^*D^4(z_2^N - z_2) \end{pmatrix},$$

where $P^N(q^N) \equiv (z_1^N, z_2^N)$, and $z_1^N = P_3^N z_1$, $z_2^N = P_2^N z_2$, where P_3^N, P_2^N are the projection operators of Lemmas 1.19 and 1.18. Thus,

$$|\mathscr{A}(q^*)(P^N(q^N)z-z)|^2 = |z_2^N - z_2|^2_{2,q_1^*} + |q_1^*D^4(z_1^N - z_1) + q_2^*D^4(z_2^N - z_2)|_0^2$$
$$\leq |z_2^N - z_2|^2_{2,q_1^*} + \{q_1^*|D^4(z_1^N - z_1)|_0$$
$$+ |q_2^*||D^4(z_2^N - z_2)|_0\}^2$$
$$\leq q_1^*\,\kappa_{2,2}^2\left(\tfrac{1}{N}\right)^8|D^6 z_2|_0^2 + \{q_1^*\,\kappa_{3,2}\left(\tfrac{1}{N}\right)^2|D^6 z_1|_0$$
$$+ |q_2^*|\kappa_{2,4}\left(\tfrac{1}{N}\right)^2|D^6 z_2|_0\}^2,$$

by an application of Lemmas 1.18 and 1.19. Thus, $|\mathscr{A}(q^*)(P^N(q^N)z-z)| \to 0$ as $N \to \infty$.

For the last term in (2.8), we have $P^N(q) \to I$ strongly on Z. Convergence of the first term follows once we write out explicitly

$$(\mathscr{A}(q^N) - \mathscr{A}(q^*))P^N(q^N)z = \begin{pmatrix} 0 \\ -(q_1^N - q_1^*)D^4 z_1^N - (q_2^N - q_2^*)D^4 z_2^N \end{pmatrix}$$

and note that $D^4 z_1^N \to D^4 z_1$ and $D^4 z_2^N \to D^4 z_2$ in H^0 and $q^N \to q^*$. Thus Trotter-Kato (Prop. 1.5) yields convergence of the semigroups $|T^N(t;q^N)z - T(t;q^*)z| \to 0$ uniformly on $[0,T]$ for each $z \in Z$. $\square$

<u>Remark</u>. Note the role that the dense subsets $\mathscr{D}$ played in the proof, in addition to possessing the required smoothness to apply Lemmas 1.18 and 1.19. Since $\mathscr{D} \subset \mathrm{Dom}(\mathscr{A}_i) \subset \mathrm{Dom}(\tilde{\mathscr{A}}_i)$, we were able to use the form of $\mathscr{A}_i$ explicitly in the proof since $\tilde{\mathscr{A}}_i$ restricted to $\mathrm{Dom}(\mathscr{A}_i)$ is $\mathscr{A}_i$ itself. Thus the Trotter-Kato approach does not require that we know $\tilde{\mathscr{A}}_i$ explicitly. $\square$

A direct application of Proposition 1.4 along with the compactness assumption (HQ) yields convergence of a subsequence of the solutions $\bar{q}^N$ of the approximate identification problem to a solution $\bar{q}$ of the original identification problem (IDA).

The case where $\gamma > 0$ in (2.2) follows easily from perturbation theory. In this case $\mathscr{A}_k$ in (2.5) becomes

$$\mathscr{C}_k = \begin{bmatrix} 0 & 1 \\ -\alpha D^4 & -\delta D^4 - \gamma \end{bmatrix}.$$

Consider $\mathscr{C}_k = \tilde{\mathscr{A}}_k + B$, where $\mathscr{A}_k = \begin{bmatrix} 0 & 1 \\ -\alpha D^4 & -\delta D^4 \end{bmatrix}$ as before, and $\tilde{\mathscr{A}}_k$ is its maximal dissipative extension, and $B = \begin{pmatrix} 0 & 0 \\ 0 & -\gamma \end{pmatrix}$. $\mathscr{L}_k$ is a bounded perturbation of $\tilde{\mathscr{A}}_k$, since $|Bz| \leq \gamma |z_2|_0$ for every $z = (z_1, z_2) \in Z$. So standard perturbation results [18, p. 33] imply that $\mathscr{L}_k$ generates a C_0 semigroup on Z. It is easily verified that $(\mathscr{L}_k - \lambda I)\mathscr{D} = \mathscr{D}$. The proof of Theorem 2.21 remains essentially unchanged by the inclusion of this new term.

We now turn to a concrete realization of the approximate identification problem. Let B_i^N be the quintic B-spline satisfying the appropriate boundary conditions, as defined in (1.16)-(1.18), and recall that $\{B_i^N\}_{i=0}^N$ are a basis for $S_k^5(\Delta^N)$. As a basis for $Z^N = S_k^5(\Delta^N) \times S_k^5(\Delta^N)$, then we take $\{\beta_i\}_{i=0}^{2N+1}$, where

$$\beta_i^N = \begin{cases} (B_i^N, 0)^T, & 0 \leq i \leq N \\ (0, B_{i-(N+1)}^N)^T, & N+1 \leq i \leq 2N+1 \end{cases}$$

Our approximating equation

$$\dot{z}^N(t) = \mathscr{A}^N z^N(t) + P^N F(t)$$

is then defined by $\mathscr{A}^N = P^N \mathscr{A} P^N$ (with $F = (0,f)^T$), where P^N is the orthogonal projection of Z onto Z^N. To obtain a realization of this method, we seek a $z^N(t) \in Z^N$ satisfying

$$\langle \dot{z}^N(t) - \mathscr{A}^N z^N(t) - P^N F(t), \zeta \rangle = 0 \quad \text{for all} \quad \zeta \in Z^N.$$

Since $z^N(t)$ and $\zeta \in Z^N$, they have representations $z^N(t) = \sum w_j^N(t) \beta_j^N$ and $\zeta = \sum v_i^N \beta_i^N$. The condition above thus reduces to

$$\langle \sum \dot{w}_j^N(t) \beta_j^N - \sum w_j^N \mathscr{A} \beta_j^N - F, \beta_i^N \rangle = 0$$

for $i = 0,1,\ldots,2N+1$, which in turn is equivalent to the vector system of ordinary differential equations

$$(2.9)\qquad \begin{aligned} Q^N\dot{w}^N(t) &= K^N w^N(t) + R^N F(t) \\ Q^N w^N(0) &= R^N \Psi \end{aligned}$$

where $w^N(t) = (w_0^N(t),\dots,w_{2N+1}^N(t))$, $Q_{ij}^N = \langle\beta_i^N,\beta_j^N\rangle$, $K_{ij}^N = \langle\beta_i^N,\mathscr{A}\beta_j^N\rangle$, $(R^N F)_j = \langle\beta_j^N,F\rangle$, and $\Psi = (\phi,\psi)$ is the vector of initial conditions in (2.4). For the examples under consideration (as well as others presented below), the matrix structure in the approximating equations facilitate computations. In particular, one finds

$$Q^N = \begin{bmatrix} Q_1^N & 0 \\ 0 & Q_2^N \end{bmatrix}$$

where $(Q_1^N)_{ij} = \alpha\langle D^2B_i^N, D^2B_j^N\rangle_0$, $(Q_2^N)_{ij} = \langle B_i^N,B_j^N\rangle_0$ and

$$K^N = \begin{bmatrix} 0 & K_1^N \\ K_2^N & K_3^N \end{bmatrix}$$

with $K_1^N = Q_1^N$, $(K_2^N)_{ij} = \langle -\alpha D^4B_i^N,B_j^N\rangle_0 = -(Q_1^N)_{ij}$, and $(K_3^N)_{ij} = \langle -\delta D^4B_i^N,B_j^N\rangle - \langle\gamma B_i^N,B_j^N\rangle_0 = -\frac{\delta}{\alpha}(Q_1^N)_{ij} - \gamma(Q_2^N)_{ij}$, where one uses integration by parts and the boundary conditions to establish these identities. Thus the equation in (2.9) reduces to

$$(2.10)\qquad \dot{w}^N(t) = G^N w^N(t) + F^N(t)$$

where

$$G^N = \begin{bmatrix} 0 & I \\ -(Q_2^N)^{-1}Q_1^N & -\frac{\delta}{\alpha}(Q_2^N)^{-1}Q_1^N - \gamma I \end{bmatrix}$$

and

$$F^N = (Q^N)^{-1}R^N F = \begin{bmatrix} 0 \\ (Q_2^N)^{-1}f^N \end{bmatrix}$$

with $f_j^N = \langle f, B_j^N\rangle_0$, $j = 0,1,\ldots,N$.

The approximate displacement $y^N(t,x)$, used in the cost functional J, is given by

$$y^N(t,\cdot) \equiv z_1^N(t) = \sum_{i=0}^{N} w_i^N(t)B_i^N.$$

The matrices Q_1^N and Q_2^N have a banded structure and can be used efficiently in solving (2.10). More will be said about this in Chapter 4.

Section 3. An Approximation Using Cubic Splines

In Section 2, we solved the approximate identification problem (IDA) using state approximations based upon quintic splines. In this section, we lower the smoothness requirement for the basis elements by rewriting (2.2) as an abstract equation which permits the use of cubic splines. To do this, we consider here only the case where δ and γ are zero (no damping), with boundary conditions of type 1, corresponding to a simply supported beam. Then we may rewrite (2.2) as an abstract equation in $Z = H^0 \times H^0 \times H^0$ of the form

$$\text{(2.11)} \qquad \begin{aligned} \dot{z}(t) &= \mathscr{A}(q)z(t) + F(q,t), \quad t > 0, \\ z(0) &= z_0 \end{aligned}$$

where $z(t) = (z_1(t),z_2(t),z_3(t)) \equiv (y(t,\cdot),y_t(t,\cdot),y_{xx}(t,\cdot))$, $z_0 = (\phi,\psi,\phi_{xx})$, and

$$\mathcal{A}(q) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\sqrt{\alpha}\ D^2 \\ 0 & \sqrt{\alpha}\ D^2 & 0 \end{bmatrix}$$

on $\mathrm{Dom}(\mathcal{A}(q)) = H^0 \times H^2 \cap H_0^1 \times H^2 \cap H_0^1$. Here $q_1 \equiv \alpha$.

We first wish to prove that $\mathcal{A}(q)$ generates a C_0 semigroup on Z. The following lemmas will do this, starting with dissipativeness of $\mathcal{A}(q)$.

<u>Lemma 2.22</u>. $(\mathcal{A}(q) - \omega I)$ is dissipative for ω sufficiently large.

<u>Proof</u>: Let $v = (v_1, v_2, v_3) \in \mathrm{Dom}(\mathcal{A}(q))$. Then,

$$\begin{aligned} <(\mathcal{A}(q)-\omega I)v,v> &= <v_2,v_1>_0 + <-\sqrt{\alpha}\ D^2 v_3, v_2> + <\sqrt{\alpha}\ D^2 v_2, v_3>_0 \\ &\quad - \omega|v_1|_0^2 - \omega|v_2|_0^2 - \omega|v_3|_0^2 \\ &= <v_2,v_1>_0 - \omega|v_1|_0^2 - \omega|v_2|_0^2 - \omega|v_3|_0^2 \\ &\le |v_2|_0|v_1|_0 - \omega|v_1|_0^2 - \omega|v_2|_0^2 - \omega|v_3|_0^2 \\ &\le \frac{|v_2|_0^2}{2} + \frac{|v_1|_0^2}{2} - \omega|v_1|_0^2 - \omega|v_2|_0^2 - \omega|v_3|_0^2 \\ &= (\tfrac{1}{2} - \omega)|v_1|_0^2 + (\tfrac{1}{2} - \omega)|v_2|_0^2 - \omega|v_3|_0^2 \\ &\le 0 \quad \text{for} \quad \omega \ge 1/2. \quad \square \end{aligned}$$

To establish that $\mathcal{A}(q)$ is a generator of a C_0 semigroup, we will show that $\mathcal{A}(q)$ is closed and $R(\mathcal{A}(q) - \lambda I) = Z$. For simplicity, we first partition the operator $\mathcal{A}(q)$ as

$$\mathscr{A}(q) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & & \\ 0 & & B(q) \end{bmatrix}$$

where

$$B(q) = \begin{bmatrix} 0 & -\sqrt{\alpha}\ D^2 \\ \sqrt{\alpha}\ D^2 & 0 \end{bmatrix}$$

on $Dom(B(q)) = H^2 \cap H_0^1 \times H^2 \cap H_0^1$, and derive some results for $B(q)$ which will be used to show that $\mathscr{A}(q)$ is a generator of a C_0 semigroup.

Lemma 2.23. $B(q)$ is a dissipative, skew-adjoint operator and generates a C_0 semigroup of contractions on $Y = H^0 \times H^0$.

Proof: Dissipativeness follows easily from an integration by parts. Let $v = (v_1,v_2) \in Dom(B(q))$; then, $\langle B(q)v,v\rangle = \langle -\sqrt{\alpha}\ D^2v_2v_1\rangle_0 + \langle\sqrt{\alpha}\ D^2v_1,v_2\rangle_0 = 0$.

It is easily verified that $-B(q)$ is adjoint to $B(q)$ $(B^* \supset -B)$:

$$\begin{aligned} \langle B(q)v,w\rangle - \langle v_2\text{-}B(q)w\rangle &= \langle -\sqrt{\alpha}\ D^2v_2,w_1\rangle_0 + \langle\sqrt{\alpha}\ D^2v_1,w_2\rangle_0 \\ &\quad - \langle v_1,\sqrt{\alpha}\ D^2w_2\rangle_0 - \langle v_2,-\sqrt{\alpha}\ D^2w_1\rangle_0 \\ &= 0 \quad \text{for all} \quad w = (w_1,w_2) \in Dom(B(q)). \end{aligned}$$

We show $B^* \subset -B$. Let $g = (g_1,g_2) \in Dom(B(q)^*)$, $f = (f_1,f_2) = B(q)^*g$. We show $f = -B(q)g$ and $g \in Dom(B(q))$. We have

$$\langle v,f\rangle = \langle v,B(q)^*g\rangle = \langle B(q)v,g\rangle \quad \text{for every} \quad v \in Dom(B(q)),$$

or,

$$\int_0^1 v_1f_1 + \int_0^1 v_2f_2 + \int_0^1 \sqrt{\alpha}(D^2v_2)g_1 - \int_0^1 \sqrt{\alpha}(D^2v_1)g_2 = 0$$

for every $v \in \mathrm{Dom}(B(q))$. Let $h_1(x) \equiv \int_0^x \int_0^{s_1} f_1(s)ds ds_1 - x \int_0^1 \int_0^{s_1} f_1(s)ds ds_1$ and $h_2(x) \equiv \int_0^x \int_0^{s_1} f_2(s)ds ds_1 - x \int_0^1 \int_0^{s_1} f_2(s)ds ds_1$. Then integrating the terms in the above equation by parts yields

$$\int_0^1 (D^2 v_1)h_1 + \int_0^1 (D^2 v_2)h_2 + \int_0^1 \sqrt{\alpha}(D^2 v_2)g_1 - \int_0^1 \sqrt{\alpha}(D^2 v_1)g_2 = 0,$$

where we have applied the conditions $v_1, v_2 \in H^2 \cap H_0^1$ and $h_i(0) = h_i(1) = 0$, for $i = 1,2$. Thus we obtain the pair of equations

$$\int_0^1 (D^2 v_1)(h_1 - \sqrt{\alpha} g_2) = 0$$

and

$$\int_0^1 (D^2 v_2)(h_2 + \sqrt{\alpha} g_1) = 0.$$

Thus $g_2 = \frac{1}{\sqrt{\alpha}} h_1$, $Dg_2 = \frac{1}{\sqrt{\alpha}} Dh_1$, $D^2 g_2 = \frac{1}{\sqrt{\alpha}} f_1 \in H^0$ implies $g_2 \in H^2$, and $g_2(0) = \frac{1}{\sqrt{\alpha}} h_1(0) = 0$, $g_2(1) = \frac{1}{\sqrt{\alpha}} h_1(1) = 0$ implies $g_2 \in H^2 \cap H_0^1$. Similarly, we obtain $g_1 \in H^2 \cap H_0^1$ and $D^2 g_1 = -\frac{1}{\sqrt{\alpha}} f_2$, and so $(f_1, f_2)^T = B(q)^* g = (\sqrt{\alpha}\, D^2 g_2, -\sqrt{\alpha} D^2 g_1)^T = -B(q)g$.

Therefore, $B(q)$ is skew-adjoint. Since $\langle B(q)^* v, v\rangle = \langle -B(q)v, v\rangle = 0$, $B(q)$ is maximal dissipative [25, p. 87], and so it generates a C_0 semigroup of contractions on Y [25, p. 88]. □

We may now state and prove the result for $\mathscr{A}$:

<u>Lemma 2.24</u>. $\mathscr{A}(q)$ is the generator of a C_0 semigroup $T(t;q)$ on $Z = H^0 \times H^0 \times H^0$.

<u>Proof</u>: Let $z = (z_1, z_2, z_3) \in \mathrm{Dom}(\mathscr{A}(q))$. Since $B(q)$ generates a semigroup on $H^0 \times H^0$, we have [25 , p. 87] $R(B(q) - I) = H^0 \times H^0$ for any $\lambda > 0$ and $B(q)$ is a closed operator. But

since

$$(\mathscr{A}(q) - \lambda I)z = \begin{bmatrix} z_2 - \lambda z_1 \\ (B(q)-\lambda I)\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} \end{bmatrix}$$

$R(B(q) - \lambda I) = H^0 \times H^0$ implies that given $f_2, f_3 \in H^0$, there exists $z_2, z_3 \in \mathrm{Dom}(B(q))$ such that $(B(q) - \lambda I)\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} f_2 \\ f_3 \end{pmatrix}$; then, given $f_1 \in H^0$, $z_1 = \frac{1}{\lambda}(z_2 - f_1) \in H^0$ and z solves $(\mathscr{A}(q) - \lambda I)z = f$, $f = (f_1, f_2, f_3)$, which implies $R(\mathscr{A}(q)-\lambda I) = Z$. Also the fact that $B(q)$ is a closed operator yields immediately that $\mathscr{A}(q)$ is a closed operator, and these two conditions along with the dissipativeness of $\mathscr{A}(q) - \omega I$ are sufficient [25, pp. 87-88] to ensure that $\mathscr{A}(q)$ generates a C_0 semigroup on Z. □

We again require a dense subset of smooth functions $\mathscr{D} \subset \mathrm{Dom}(\mathscr{A}(q))$. Note that for $\phi_j(x) = \sqrt{2}\sin(j\pi x)$, $\{\phi_j\}_{j=1}^{\infty}$ forms a complete orthonormal set for H^0. Define $\gamma_{1,j} = \begin{bmatrix} \phi_j \\ 0 \\ 0 \end{bmatrix}$, $\gamma_{2,j} = \begin{bmatrix} 0 \\ \phi_j \\ 0 \end{bmatrix}$, and $\gamma_{3,j} = \begin{bmatrix} 0 \\ 0 \\ \phi_j \end{bmatrix}$. Then define

$\mathscr{D}^N = \mathrm{span}\{\gamma_{1,j}\}_{j=1}^{N} \cup \mathrm{span}\{\gamma_{2,j}\}_{j=1}^{N} \cup \mathrm{span}\{\gamma_{3,j}\}_{j=1}^{N}$, and $\mathscr{D} = \bigcup_{N=1}^{\infty} \mathscr{D}^N$. then $\bar{\mathscr{D}} = Z$. For $\lambda > 0$, it is easily verified that $(\lambda I - \mathscr{A}(q))\mathscr{D} = \mathscr{D}$, and so $\overline{(\lambda I - \mathscr{A}(q))\mathscr{D}} = Z$.

For the approximate identification problem, define $Z^N = S_0^3(\Delta^N) \times S_0^3(\Delta^N) \times S_0^3(\Delta^N)$, and let P^N be the orthogonal projection $P^N \colon Z \longrightarrow Z^N$. In this case, for $z = (z_1, z_2, z_3) \in Z$, $P^N z = (P_0^N z_1, P_0^N z_2, P_0^N z_3)$ where P_0^N is the orthogonal projection of H^0 onto $S_0^3(\Delta^N)$. The approximate identification problem then corresponds to solutions of

$$(2.12)\qquad \begin{aligned} \dot z^N(t) &= \mathscr{A}^N(q)\, z^N(t) + P^N F(q,t), \quad t > 0 \\ z^N(0) &= z_0^N \end{aligned}$$

where $\mathscr{A}^N(q) = P^N \mathscr{A}(q) P^N$ and $z_0^N = P^N z_0$.

We defer obtaining a concrete realization of this problem to the end of this section, after we have proved convergence of a subsequence of the solutions $\bar q^N$ of the approximate identification problem converge to a solution $\bar q$ of the original identification problem.

Lemma 2.25. Assuming (HQ), the projections P^N converge strongly to the identity I in Z.

Proof: Let $(z_1, z_2, z_3) \in \mathscr{D}$, where $\mathscr{D}$ is the dense subset $\mathscr{D} \subset \mathrm{Dom}(\mathscr{A}(q)) \subset Z$ as defined above. Then, $P^N z = (P_0^N z_1, P_0^N z_2, P_0^N z_3) \equiv (z_1^N, z_2^N, z_3^N)$, where P_0^N is the $|\cdot|_0$-projection of Z onto $S_0^3(\Delta^N)$. Then,

$$\begin{aligned} |P^N z - z|^2 &= \sum_{i=1}^{3} |P_0^N z_i - z_i|_0^2 \\ &\le \sum_{i=1}^{3} \left[\kappa_{0,0}\left(\frac{1}{N^4}\right) |D^4 z_i|_0\right]^2, \text{ by Lemma 1.5.} \end{aligned}$$

Therefore $P^N \longrightarrow I$ on $\mathscr{D}$, and the boundedness of the projection operator implies, since $\bar{\mathscr{D}} = Z$, that $P^N \longrightarrow I$ strongly on Z. □

We now state and prove the convergence result for cubic spline approximations.

Theorem 2.26. Let (HQ) hold. Then the semigroups $T(t;q)$ and $T^N(t;q)$ generated by $\mathscr{A}(q)$ and $\mathscr{A}^N(q)$ satisfy $\|T(t;q)\| \le Me^{\omega t}$ and $\|T^N(t;q)\| \le Me^{\omega t}$. Moreover, for any sequence $\{q^N\}$ converging to q^* in Q, we have

$$|T^N(t;q^N)z - T(t;q^*)z| \longrightarrow 0 \text{ uniformly on } [0,T]$$

for each $z \in Z$.

Proof: The bound $\|T(t;q)\| \le Me^{\omega t}$ follows because $\mathscr{A}(q)$ generates a C_0 semigroup on Z [31, p. 10]. Since $\langle \mathscr{A}^N(q)z,z\rangle = \langle \mathscr{A}(q)P^Nz,P^Nz\rangle = \omega|P^Nz| \le \omega|z|$, by the dissipativeness of $\mathscr{A} - \omega I$, for ω sufficiently large, we obtain $\|T^N(t;q)\| \le Me^{\omega t}$.

We refer again to the Trotter-Kato theorem (Proposition 1.5) to establish convergence. Let $\mathscr{B} = Z$, $\mathscr{B}^N = Z^N$, and take π^N to be the identity. All that remains to be verified is condition (iii) of Trotter-Kato.

Taking $\mathscr{D}$ as above, we see that for each $z = (z_1,z_2,z_3) \in \mathscr{D}$,

$$\begin{aligned} |\mathscr{A}^N(q^N)z - \mathscr{A}(q^*)z| &= |P^N\mathscr{A}(q^N)P^Nz - \mathscr{A}(q^*)z| \\ &\le |(\mathscr{A}(q^N) - \mathscr{A}(q^*))P^Nz| \\ &\quad + |\mathscr{A}(q^*)(P^Nz-z)| + |(P^N-I)\mathscr{A}(q^*)z|. \end{aligned} \tag{2.13}$$

We bound each of the terms above. The second term may be written explicitly as

$$\mathscr{A}(q^*)(P^Nz-z) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\sqrt{\alpha}\, D^2 \\ 0 & \sqrt{\alpha}\, D^2 & 0 \end{pmatrix}(P^Nz - z)$$

$$= \begin{pmatrix} z_2^N - z_2 \\ -\sqrt{\alpha}D^2(z_3^N - z_3) \\ \sqrt{\alpha}D^2(z_2^N - z_2) \end{pmatrix}$$

where $q_1^* = \alpha$ and $z_i^N = P_0^N z_i$, $i = 1,2,3$. Thus,

$$|\mathscr{A}(q^*)(P^N z - z)|^2 = |z_2^N - z_2|_0^2 + \alpha|D^2(z_3^N - z_3)|_0^2 + \alpha|D^2(z_1^N - z_1)|_0^2,$$

which implies, by Lemma 1.15,

$$|\mathscr{A}(q^*)(P^N z - z)|^2 \le \left[\kappa_{0,0}(\tfrac{1}{N})^4|D^4 z_2|_0\right]^2 + \alpha\left[\kappa_{0,2}(\tfrac{1}{N})^2|D^4 z_3|_0\right]^2$$
$$+ \alpha\left[\kappa_{0,2}(\tfrac{1}{N})^2|D^4 z_2|_0\right]^2,$$

and so $|\mathscr{A}(q^*)(P^N z - z)| \to 0$ as $N \to \infty$.

For the last term of (2.13), we have $P^N \longrightarrow I$ strongly on Z. Convergence of the first term follows explicitly from the form of $\mathscr{A}(q)$:

$$(\mathscr{A}(q^N) - \mathscr{A}(q^*))P^N z = \begin{bmatrix} 0 \\ -(\sqrt{q_1^N} - \sqrt{q_1^*})D^2 z_3^N \\ -(\sqrt{q_1^*} - \sqrt{q_1^N})D^2 z_2^N \end{bmatrix}.$$

Convergence follows since $D^2 z_2^N \longrightarrow D^2 z_2$ and $D^2 z_3^N \longrightarrow D^2 z_3$ in H^0, and by hypothesis $q^N \longrightarrow q^*$. $\square$

This theorem, together with Proposition 1.4, is sufficient to ensure that

$$\lim_{N\to\infty}|q^N - q^*| = 0 \quad \text{implies} \quad \lim_{N\to\infty}|z^N(t;q^N) - z(t;q^*)| = 0.$$

We now turn to a concrete realization of the approximate problem using cubic splines. Let $\{C_i^N\}$ be the cubic splines defined in (1.15) which span $S_0^3(\Delta^N)$. The approximation subspace Z^N is $S_0^3(\Delta^N) \times S_0^3(\Delta^N) \times S_0^3(\Delta^N)$. We follow the procedure outlined at the end of Section 2 where we obtained a concrete realization using quintic splines.

We form projections $P^N: Z \to Z^N$ and seek approximate solutions of the form $z^N(t) = \sum_{i=0}^{N} w_i^N(t)\xi_i^N$ where $\xi_i^N \in Z^N$ is given by

$$\xi_i^N = \begin{cases} (C_i^N,0,0)^T & i = 0,\dots,N, \\ (0,C_{i-(N+1)}^N,0)^T & i = N+1,\dots,2N+1, \\ (0,0,C_{i-(2N+2)}^N)^T & i = 2N+2,\dots,3N+2. \end{cases}$$

We are thus led to a system of $3N+3$ differential equations for the w_i^N (compare with the $2N+2$ system in the quintic formulation!)

$$\dot{w}^N(t) = G^N w^N(t) + F^N(t) \tag{2.14}$$

where

$$G^N = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & -\alpha(A_2^N)^{-1}A_1^N \\ 0 & (A_2^N)^{-1}A_1^N & 0 \end{bmatrix}$$

with

$$(A_2^N)_{i,j} = \langle C_i^N, C_j^N\rangle_0, \quad (A_1^N)_{i,j} = \langle DC_i^N, DC_j^N\rangle_0, \quad F^N = (Q^N)^{-1}R^N F$$

$$(Q^N) = \mathrm{diag}(A_2^N, A_2^N, A_2^N)$$

and

$$(R^N F)_i = \begin{cases} 0 & 0 \le i \le N,\ 2N+2 \le i \le 3N+2 \\ \langle f, C_{i-(N+1)}^N\rangle_2 & N+1 \le i \le 2N+1. \end{cases}$$

Finally, we present an approximate method using cubic splines in the case where structural damping is included. While the convergence proof falls outside the framework above, we present numerical results in Section 4 which demonstrate the effectiveness of this method. We next consider (2.2) with $\gamma = 0$, $\alpha > 0$, $\delta > 0$ and boundary conditions of type 1 (simply supported). The initial conditions are the same as above. Equation (2.2) is rewritten as

$$\begin{aligned} \dot{y} &= v \\ \delta D^2\dot{u} + \dot{v} &= -\alpha D^2 u \\ \dot{u} &= D^2 v \end{aligned}$$

so that for $z = (y,v,u)^T = (y,y_t,y_{xx})^T$ in $Z = H^0 \times H^0 \times H^0$, we obtain the abstract equation

$$\text{(2.15)} \qquad \begin{aligned} \Gamma\dot{z}(t) &= \mathscr{A}z(t) \\ z(0) &= (\phi,\psi,\phi'')^T. \end{aligned}$$

Here $\mathscr{A}$ is given by

$$\mathscr{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -\alpha D^2 \\ 0 & D^2 & 0 \end{pmatrix}$$

on $\mathrm{Dom}(\mathscr{A}) = (H^2 \cap H_0^1) \times (H^2 \cap H_0^1) \times (H^2 \cap H_0^1)$, and

$$\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \delta D^2 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that only if $\delta = 0$ (no structural damping) does this reduce to the form (2.11) and fall within the framework of the theory outlined for (2.11) and (2.12). Nonetheless, the general ideas discussed above lead to efficient computational schemes for (2.11). Taking $Z^N = S_0^3(\Delta^N) \times S_0^3(\Delta^N) \times S_0^3(\Delta^N)$ and P^N the orthogonal projection of Z onto Z^N, we define $\mathscr{A}^N = P^N\mathscr{A}P^N$ and $\Gamma^N = P^N\Gamma P^N$ and use

$$(2.16) \qquad \Gamma^N \dot{z}^N(t) = \mathscr{A}^N z^N(t)$$

as the approximating equation for (2.14).

Section 4. Numerical Results

We provide a few examples of numerical experiments illustrating the methods presented in this chapter. Many examples have already been reported in [6] which compare the method based upon quintic splines to the method based upon cubic splines for the Euler-Bernoulli equation with structural damping. Included were examples with time-dependent boundary conditions which were transformed to homogeneous boundary conditions. We refer the interested reader to those examples and do not repeat them here.

We provide some new examples not included in [6]. The first example concerns a simply supported beam with both structural and viscous damping. The others illustrate the convergence properties for the important case of the cantilever beam. All of these examples use the quintic spline approximation.

The numerical experiments consisted of taking as data the values of a solution of a model equation of the form (2.1) whose parameters were known and then seeking a solution $\bar{q}^N$ of the approximate identification problem for different values of N. The "data" $\{y_{ij}\}$, $i = 1,\dots,r$; $j = 1,\dots,\ell$ were generated by solving (2.1) approximately by a Crank-Nicolson scheme and using $\hat{y}_{ij} = y(t_i,x_j)$ as the observations or input to the approximate identification packages. For the experiments listed below, we used $\ell = 3$, $r = 10$ and generated the "observations" at $x_j = j/4$, $j = 1,\dots,3$ for times $t_i = i/10$, $i = 1,\dots,10$.

The optimization algorithm (see Chapter 4) requires an initial guess for the parameters; these are referred to as the "start up" values in the tables. The values to which the optimization algorithm converged for a fixed N are denoted by $\bar{q}^N$, and $J(\bar{q}^N)$ denotes the cost functional evaluated at $\bar{q}^N$.

Example 2.1. We consider an example where we have included both structural and viscous damping in the model. We consider the model equation

$$y_{tt} = -q_1 y_{xxxx} - q_2 y_{xxxxt} - q_3 y_t + e^x \sin t$$

$$y(0,x) = y_t(0,x) = 0$$

$$y(t,0) = y_{xx}(t,0) = y(t,1) - y_{xx}(t,1) = 0$$

Table 2.1 summarizes the results for the parameter estimation problem for this equation using the quintic spline approximation.

Table 2.1

N	q_1^N	q_2^N	q_3^N	$J(q^N)$
2	.999938	.009875	.032420	$.2001 \times 10^{-5}$
3	.999940	.009974	.022713	$.8231 \times 10^{-7}$
4	.999936	.010004	.019860	$.3324 \times 10^{-7}$
TRUE VALUE	1.0	.01	.02	
START UP	.65	.005	.005	

Example 2.2. Next we consider an example for a cantilevered beam. With $\phi(x) = \sin ax - \sinh ax + K(\cosh ax - \cos ax)$ where $K = (\sin a + \sinh a)/(\cos a + \cosh a)$ and $a \simeq 7.8547$ (corresponding to the third mode for an undamped cantilevered beam), we

consider the equation

$$y_{tt} = -q_1 y_{xxxx} - q_2 y_{xxxxt}$$

$$y(0,x) = \phi(x), \quad y_t(0,x) = 0$$

$$y(t,0) = y_x(t,0) = y_{xx}(t,1) = y_{xxx}(t,1) = 0.$$

The numerical results are summarized in Table 2.2.

Table 2.2

$\underline{N}$	$\bar{q}_1^N$	$\bar{q}_2^N$	$J(\bar{q}^N)$
2	.3821	.0067	$.025 \times 10^{-1}$
3	.4960	.0098	$.735 \times 10^{-4}$
4	.4996	.010020	$.116 \times 10^{-6}$
6	.4997	.009995	$.768 \times 10^{-7}$
TRUE VALUE	.5	.01	
START UP	.35	.005	

Example 2.3. A second example for a cantilevered beam is provided for the model equation

$$y_{tt} = -q_1 y_{xxxx} - q_2 y_{xxxxt} + q_{10} \exp[-20(1-x)]\exp[-q_9 t], \ t > 0$$

$$y(0,x) = y_t(0,x) = 0$$

$$y(t,0) = y_x(t,0) = y_{xx}(t,1) = y_{xxx}(t,1) = 0.$$

Table 2.3 summarizes the results for the estimation problem corresponding to this equation using the quintic spline approximation.

Table 2.3

N	q_1^N	q_2^N	q_9^N	q_{10}^N	$J(q^N)$
2	.497863	.009923	2.02101	10.0458	$.110 \times 10^{-5}$
3	.498527	.009995	2.01284	10.0255	$.693 \times 10^{-7}$
4	.499284	.009996	2.00602	10.0143	$.449 \times 10^{-8}$
8	.499775	.009982	2.00183	10.0075	$.356 \times 10^{-9}$
TRUE VALUE	.5	.01	2.0	10.	
START UP	.35	.005	1.5	8.0	

When q_9 and q_{10} were treated as known values and optimization was performed for q_1 and q_2 only, the following results were obtained.

Table 2.4

N	q_1^N	q_2^N	$J(q^N)$
2	.500394	.009445	$.146 \times 10^{-5}$
3	.500045	.009763	$.141 \times 10^{-6}$
4	.500006	.009817	$.340 \times 10^{-7}$
TRUE VALUE	.5	.01	
START UP	.35	.005	

We note that in all of the examples considered for the Euler-Bernoulli equation rapid convergence of $\bar{q}^N$ to the true valued occurred.

Fc the optimization algorithm (see Chapter 4), we specified small convergence tolerances to obtain best possible results. For a typical problem, convergence was obtained in 3-5 iterations of the Levenberg-Marquardt algorithm.

A variable step/variable order method (DGEAR) was used to integrate the approximating system of ordinary differential equations. For example 2.2 with a requested local error tolerance of $1. \times 10^{-6}$, DGEAR took 211 steps (last stepsize was .018) and used fifth order methods to integrate the system when stiff methods were used. By comparison, when non-stiff methods were chosen, it took 5766 step (last stepsize was $.213 \times 10^{-3}$) to achieve the same local error and it primarily used order 2 methods, indicating a moderate degree of stiffness due to the damping term.

As was the case with the second order equations considered in [7], the $P^N \otimes P^N$ approximations for linear constant coefficient problems were extremely accurate, even for small N. It is expected that the power of these approximations will be more important for other problems to be considered in the future, particularly the case of spatially varying coefficients.

CHAPTER 3. APPLICATION TO THE TIMOSHENKO EQUATION

Section 1. The Timoshenko Equation

The second example of the application of these techniques that we wish to consider is the identification problem associated with the Timoshenko equations for transverse vibrations of beams. The Timoshenko theory extends the Euler-Bernoulli theory by taking into account the effects of rotary intertia and shear distortion. These effects play a significant role when the depth of the beam is large when compared to its span and when high frequency oscillations must be considered.

While the Timoshenko formulation may be written as a single fourth order partial differential equation, it is easier to handle boundary conditions when it is written as a system of two partial differential equations in the transverse displacement $y(t,x)$ and angle of rotation $\psi(t,x)$ of the beam cross-section from its original vertical position [14, p. 300]:

$$
(3.1) \qquad \begin{aligned} y_{tt} &= a(y_{xx} - \psi_x) + f(t,x;q) \\ \psi_{tt} &= c\psi_{xx} + b(y_x - \psi), \end{aligned} \qquad t > 0,\ x \in [0,1]
$$

where $a = k'AG/m$, $b = Aa/I$, $c = EI/m$, with A = cross sectional area of beam, E = Young's modulus, G = shear modulus, I = moment of inertia, and k' = shear coefficient (cf. [15]), and where $f(t,x;q)$ is the "load" or applied force.

The initial conditions are

$$y(0,x) = y_0(x)$$
$$y_t(0,x) = y_1(x)$$
$$\psi(0,x) = \psi_0(x)$$
$$\psi_t(0,x) = \psi_1(x),$$

and some of the common boundary conditions are

i) $y(t,0) = y(t,1) = \psi(t,0) = \psi(t,1) = 0$

ii) $y(t,0) = y(t,1) = \frac{\partial\psi}{\partial x}(t,0) = \frac{\partial\psi}{\partial x}(t,1) = 0$

iii) $y(t,0) = \psi(t,0) = \frac{\partial\psi}{\partial x}(t,1) = 0, \quad \frac{\partial y}{\partial x}(t,1) = \psi(t,1).$

These correspond to the boundary conditions for a beam fixed at both ends (i), a simple supported beam (ii), and a cantilevered beam (iii), comparable to the boundary conditions of type k for the Euler-Bernoulli equation (cf. [13, p. 97]).

For the ease of exposition, we shall limit our discussion to boundary conditions (i), although the theory is generally applicable to all three.

Section 2. An Approximation Using Cubic Splines

Following again the approach outlined in Chapter 1, we write (3.1) as an abstract equation

$$\begin{aligned} \dot{z}(t) &= \mathscr{A}(q)z(t) + F(q,t) \\ z(0) &= z_0(q) \end{aligned} \tag{3.2}$$

in a Hilbert space Z, where $q_1 = a$, $q_2 = b$, $q_3 = c$, and $q_4,\ldots,q_p$ are parameters in the load term F and initial function z_0.

Again, there are many choices of (3.2) and the space Z which lead to well-posed problems. The forms that we choose for (3.2) lead to specific state approximations. We shall discuss two such possibilities and discuss the approximate identification problem of the first in detail.

The first such choice of the abstract equation will be (3.2) with $z(t) = (z_1(t),z_2(t),z_3(t),z_4(t))^T = (y(t,\cdot),y_t(t,\cdot),\psi(t,\cdot),\psi_t(t,\cdot))$, in $Z = V(a) \times H^0 \times V(c) \times H^0$, and with

$$(3.3) \qquad \mathscr{A}(q) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ aD^2 & 0 & -aD & 0 \\ 0 & 0 & 0 & 1 \\ bD & 0 & cD^2-b & 0 \end{bmatrix}$$

on $\mathrm{Dom}(\mathscr{A}) = H^2 \cap H_0^1 \times H_0^1 \times H^2 \cap H_0^1 \times H_0^1$, where $V(\alpha) = H_0^1(\alpha)$ equipped with the inner product $\langle u,v\rangle_{V(\alpha)} \equiv \langle u,v\rangle_{1,\alpha} \equiv \langle \alpha Du,Dv\rangle_0$. We denote the corresponding norm on $V(\alpha)$ as $|\cdot|_{1,\alpha}$, and take $F(q,t) = (0,f(t,\cdot;q),0,0)^T$.

<u>Theorem 3.1</u>. The operator $\mathscr{A}(q)$ defined by (3.3) with $q_1 = a$, $q_2 = b$, $q_3 = c$ is the generator of C_0 semigroup $T(t;q)$ on Z satisfying $\|T(t;q)\| \le e^{\omega t}$ for some $\omega > 0$.

<u>Proof</u>: Define the operator $\mathscr{A}_0$ on Z by

$$\mathscr{A}_0(q) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ aD^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & cD^2 & 0 \end{bmatrix} = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

where $B_1 = \begin{bmatrix} 0 & 1 \\ aD^2 & 0 \end{bmatrix}$ on $\mathrm{Dom}(B_1) = H^2 \cap H_0^1 \times H_0^1$ and $B_2 = \begin{bmatrix} 0 & 1 \\ cD^2 & 0 \end{bmatrix}$ on $\mathrm{Dom}(B_2) = H^2 \cap H_0^1 \times H_0^1$. B_1 and B_2 are the wave operators considered in [7, p. 29], where they were shown to be generators of C_0 semigroups of contractions in $V(q_i) \times H^0$ for $i = 1$ and $i = 3$ respectively. Thus $\mathscr{A}_0$ is the generator of a C_0 semigroup of contractions $T_0(t;q)$ in Z, $\mathscr{A}_0$ is maximal dissipative, and $\|T_0(t;q)\| \leq 1$.

Now we apply perturbations results to obtain the conclusion of the theorem. We first show that $\mathscr{A}$ is a bounded perturbation of the operator $\mathscr{A}_0$. Note that $\mathscr{A} = \mathscr{A}_0 + \tilde{\mathscr{A}}$ where

$$\tilde{\mathscr{A}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -aD & 0 \\ 0 & 0 & 0 & 0 \\ bD & 0 & -b & 0 \end{pmatrix}.$$

$$\begin{aligned}
|\tilde{\mathscr{A}} z| &= |-aDz_3|_0 + |bDz_1 - bz_3|_0 \\
&\leq |aDz_3|_0 + |bDz_1|_0 + |bz_0|_0 \\
&= \frac{c}{a}|z_3|_{1,c} + \frac{b}{a}|z_1|_{1,a} + b|z_3|_0, \quad \text{by definition of } Z, \\
&\leq \frac{c}{a}|z_3|_{1,c} + \frac{b}{a}|z_1|_{1,a} + \frac{b}{\pi}|Dz_3|_0, \quad \text{by the Rayleigh-Ritz inequality,} \\
&= \frac{c}{a}|z_3|_{1,c} + \frac{b}{a}|z_1|_{1,a} + \frac{b}{\pi c}|z_3|_{1,c} \\
&= K_1|z_1|_{1,a} + K_2|z_3|_{1,c}, \quad \text{where } K_1 = \frac{b}{a} \text{ and } K_2 = \frac{c}{a} + \frac{b}{\pi c} \\
&\leq K_3|z| \quad \text{with } K_3 = \max(K_1, K_2).
\end{aligned}$$

Thus $\mathscr{A}$ is a bounded perturbation of $\mathscr{A}_0$, and we can apply the perturbation results of [31, p. 80] to infer that $\mathscr{A}$ generates a C_0 semigroup on Z, satisfying

$$\|T(t;q)\| \leq e^{\|\tilde{\mathscr{A}}\| t} = e^{K_3 t}. \qquad \square$$

Corollary 3.2. The operator $(\mathscr{A} - \omega I)$ is dissipative for all ω sufficiently large.

Proof: Let $z \in \mathrm{Dom}(\mathscr{A})$. Then

$$\begin{aligned} \langle \mathscr{A}z, z\rangle &= \langle(\mathscr{A}_0 + \tilde{\mathscr{A}})z, z\rangle \\ &= \langle \mathscr{A}_0 z, z\rangle + \langle\tilde{\mathscr{A}}z, z\rangle \\ &\leq \langle\tilde{\mathscr{A}}z, z\rangle, \text{ since } \mathscr{A}_0 \text{ is dissipative,} \\ &\leq |\tilde{\mathscr{A}}z|\ |z| \\ &\leq K_3|z|^2 \text{ by the boundedness of } \tilde{\mathscr{A}}. \end{aligned}$$

Thus $\langle(\mathscr{A} - \omega I)z, z\rangle \leq 0$ for all $\omega \geq K_3$.

The proof of the above theorem is interesting in that it states that the Timoshenko equations can be viewed as a perturbation of two simple wave equations. Because of this fact, we were able to apply the results of [7] where the wave equation involving an operator of the form B_1 was treated.

Corollary 3.3. Let $\mathscr{D} = \mathrm{Dom}(\mathscr{A}) \cap (H^m \times H^m \times H^m \times H^m)$, $m \geq 3$. Then $\overline{(\mathscr{A} - \lambda_0 I)\mathscr{D}} = Z$ for some λ_0 sufficiently large.

Proof: Since $\mathscr{A}$ generates a C_0 semigroup on Z, $R(\mathscr{A} - \lambda_0 I) = Z$ for λ_0 sufficiently large. So for such λ_0 in the resolvent set $\rho(\mathscr{A})$, $\mathscr{A} - \lambda_0 I$ is invertible.

Consider $Y \subset Z$. $Y = (H^m \times H^{m-2} \times H^m \times H^{m-2}) \cap Z$. Clearly $\mathscr{A} - \lambda_0 I$ is invertible on Y, so take $f = (f_1, f_2, f_3, f_4) \in Y$. We show there exists a $z \in \mathscr{D}$ such that

$$(3.4) \qquad (\mathscr{A} - \lambda_0 I)z = f.$$

We know by the invertibility of $\mathscr{A} - \lambda_0 I$ that there exists a

$z \in \mathrm{Dom}(\mathscr{A}) = H_0^1 \cap H^2 \times H_0^1 \times H_0^1 \cap H^2 \times H_0^1$ such that (3.4) holds. We need only show $z \in \mathscr{D}$. But,

$$(\mathscr{A} - \lambda_0 I)z = \begin{pmatrix} -\lambda_0 z_1 + z_2 \\ -\lambda_0 z_2 + aD^2 z_1 - aDz_3 \\ -\lambda_0 z_3 + z_4 \\ -\lambda_0 z_4 + bDz_1 + cD^2 z_3 - z_3 \end{pmatrix}$$

So, $(\mathscr{A} - \lambda_0 I)z = f$ if and only if

(i) $-\lambda_0 z_1 + z_2 = f_1$

(ii) $-\lambda_0 z_2 + aD^2 z_1 - aDz_3 = f_2$

(iii) $-\lambda_0 z_3 + z_4 = f_3$

(iv) $-\lambda_0 z_4 + bDz_1 + cD^2 z_3 - cz_3 = f_4$.

First consider $m = 3$. Then, $f \in Y = (H^3 \times H^1 \times H^3 \times H^1) \cap Z$ implies $f_2 \in H^1$, and $z \in \mathrm{Dom}(\mathscr{A})$ implies $z_2 \in H_0^1$, $z_3 \in H^2 \cap H_0^1$ (hence $Dz_3 \in H^1$). So $g \equiv aDz_3 + \lambda_0 z_2 + f_2 \in H^1$ and (ii) implies $D^2 z_1 = g/a \in H^1$ and $z_1 \in H^3$. Then (i) implies $z_2 \in H^3$. Similarly, $f \in Y$ implies $f_4 \in H^1$ and $z \in \mathrm{Dom}(\mathscr{A})$ implies $z_1 \in H^2 \cap H_0^1$, $z_4 \in H_0^1$ and so $g \equiv (\lambda_0 z_4 - bDz_1 + cz + f_4) \in H^1$ and so $D^2 z_3 = g/c \in H^1$, which implies $z_3 \in H^3$. And then (iii) implies $z_4 \in H^3$.

Next consider $m = 4$. We have $z \in (H^3 \times H^3 \times H^3 \times H^3) \cap \mathrm{Dom}(\mathscr{A})$ from above, and we assume $f \in (H^4 \times H^2 \times H^4 \times H^2) \cap Z$. Repeating the arguments in the last paragraph yields $D^2 z_1 \in H^2$ which implies $z_1 \in H^4$ and $z_2 \in H^4$ (from (i)). Likewise, $D^2 z_3 \in H^2$ implies $z_3 \in H^4$ and $z_2 \in H^4$.

A simple induction argument yields the result for arbitrary m. [Note: we actually only use $m = 4$]. So, $(\mathscr{A} - \lambda_0 I)\mathscr{D} = Y$, and $\overline{(\mathscr{A} - \lambda_0 I)\mathscr{D}} = \overline{Y} = Z$ since H^m is dense in H^{m-j}, $j = 1,\ldots,m$. □

We may now discuss the state approximations which we use to solve the approximate identification problem. We take $Z^N = S_0^3(\Delta^N) \times S_0^3(\Delta^N) \times S_0^3(\Delta^N) \times S_0^3(\Delta^N)$, where we recall that $S_0^3(\Delta^N)$ is the set of cubic splines $s(x)$ with knots on Δ^N satisfying $s(0) = s(1) = 0$. Define $P_1^N(\alpha)$ the projection $P_1^N(\alpha): V(\alpha) \longrightarrow S_0^3(\Delta^N)$ in the $|\cdot|_{1,\alpha}$ norm, and P_0^N to be the projection $P_0^N: H^0 \longrightarrow S_0^3(\Delta^N)$ in the H^0 norm. Then it follows immediately from Lemmas 1.15 and 1.16 that with P^N defined as the projection $P^N: Z \longrightarrow Z^N$ given by $P^N z = (P_1^N z_1, P_0^N z_2, P_1^N z_3, P_0^N z_4)$, that (see also [7, p. 33]) the following holds.

Lemma 3.4. $P^N \longrightarrow I$ strongly on Z.

With Z^N and P^N defined as above, we solve the approximate identification problem associated with

$$(3.4)\qquad \begin{aligned} \dot{z}^N(t) &= \mathscr{A}^N(q) z^N(t) + F^N(q,t), \quad t > 0 \\ z^N(0) &= P^N z_0 \end{aligned}$$

in Z^N, where again we take

$$\begin{aligned} \mathscr{A}^N &= P^N \mathscr{A} P^N, \\ F^N &= P^N F. \end{aligned}$$

The operator $\mathscr{A}^N(q)$ defined on the finite dimensional subspaces Z^N is a bounded operator for each N and hence generates a C_0-semigroup $T^N(t;q) = e^{\mathscr{A}^N(q)t}$ on Z. By Theorem 3.1, $\mathscr{A}(q)$ is the generator of a C_0 semigroup satisfying $\|T(t;q)\| \leq e^{\omega t}$, so that $\mathscr{A}(q) - \omega I$ is the generator of a semigroup of contractions and is maximal dissipative [25, p. 90]. Thus by the remarks in Chapter 1, Section 2, $\mathscr{A}^N(q) - \omega I$ generates a C_0 semigroup of contractions, or $\mathscr{A}^N(q)$ generates a C_0 semigroup $T^N(t;q)$ satisfying $\|T^N(t;q)\| \leq e^{\omega t}$.

We can now state and prove the convergence result that the solutions of the approximate identification problem using state approximations (3.4) converge to the solution of the identification problem corresponding to (3.2 - 3.3).

<u>Theorem 3.5.</u> Assume (HQ) holds. Then $\mathscr{A}(q)$ and $\mathscr{A}^N(q)$ are generators of C_0 semigroups satisfying $\|T(t;q)\| \leq e^{\omega t}$ and $\|T^N(t;q)\| \leq e^{\omega t}$ for some $\omega > 0$. Moreover, for any sequence $\{q^N\}$ converging to q^* in Q, we have

$$|T^N(t;q^N)z - T(t;q^*)z| \longrightarrow 0 \text{ uniformly on } [0,T]$$

for every $z \in Z$.

<u>Proof</u>: We have already proved the first assertion. We invoke the Trotter-Kato theorem once more to prove convergence. Let $\mathscr{B} = Z(q^*)$, $\mathscr{B}^N = Z^N(q^N)$, and let $\pi^N\colon Z(q^*) \longrightarrow Z(q^N)$ be the canonical isomorphism between $Z(q^*)$ and $Z(q^N)$. Then $q^N \longrightarrow q^*$ implies $|\pi^N z| \longrightarrow z$, so that the first hypothesis of Trotter-Kato is satisfied. Thus, it remains to verify hypothesis (iii). We have already defined the set $\mathscr{D}$ such that $(\mathscr{A}(q^*) - \lambda I)\mathscr{D}$ is dense in $\mathscr{B}$.

To prove that the last hypothesis holds, let $z = (z_1, z_2, z_3, z_4) \in \mathscr{D}$. Again, we shall suppress the notation π^N, and obtain

$$
\begin{aligned}
|\mathscr{A}^N(q^N)z - \mathscr{A}(q^*)z| &= |P^N(q^N)\mathscr{A}(q^N)P^N(q^N) - \mathscr{A}(q^*)z| \\
&\leq |(\mathscr{A}(q^N) - \mathscr{A}(q^*))P^N(q^N)z| \\
&\quad + |\mathscr{A}(q^*)(P^N(q^N)z - z)| \\
&\quad + |(P^N(q^N) - I)\mathscr{A}(q^*)z|.
\end{aligned} \tag{3.5}
$$

We estimate each of these terms separately. The second term in (3.5) can be written explicitly as

$$
\mathscr{A}(q^*)(P^N(q^N)z - z) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ q_1^* D^2 & 0 & -q_1^* D & 0 \\ 0 & 0 & 0 & 1 \\ q_2^* D & 0 & q_3^* D^2 - q_2^* & 0 \end{pmatrix} (P^N(q^N)z - z)
$$

$$
= \begin{pmatrix} z_2^N - z_2 \\ q_1^* D^2(z_1^N - z_1) - q_1^* D(z_3^N - z_3) \\ z_4^N - z_4 \\ q_2^* D(z_1^N - z_1) \quad q_3^* D^2(z_3^N - z_3) - q_2^*(z_3^N - z_3) \end{pmatrix}
$$

where we have denoted $P^N(q^N)z \equiv (z_1^N, z_2^N, z_3^N, z_4^N)$. Thus,

$$
\begin{aligned}
|\mathscr{A}(q^*)(P^N(q^N)z - z)|^2 &= |z_2^N - z_2|_{1,q_1^*}^2 + |q_1^* D^2(z_1^N - z_1) - q_1^* D(z_3^N - z_3)|_0^2 \\
&\quad + |z_4^N - z_4|_{1,q_3^*}^2 + |q_2^* D(z_1^N - z_1) + q_3^* D^2(z_3^N - z_3) - q_2^*(z_3^N - z_3)|_0^2 \\
&\leq |z_2^N - z_2|_{1,q_1^*}^2 + \{q_1^* |D^2(z_1^N - z_1)|_0 + q_1^* |D(z_3^N - z_3)|_0\}^2 \\
&\quad + |z_4^N - z_4|_{1,q_3^*}^2 + \{|q_2^*| |D(z_1^N - z_1)|_0 + q_3^* |D^2(z_3^N - z_3)|_0 \\
&\qquad + |q_2^*| |z_3^N - z_3|_0\}^2
\end{aligned}
$$

$$\leq (\sqrt{q_1^*}\ \kappa_{1,0}(\tfrac{1}{N})^3|D^4z_2|_0)^2 + \{q_1^*\ \kappa_{0,2}(\tfrac{1}{N})^2|D^4z|_0$$

$$+ q_1^*\ \kappa_{0,1}(\tfrac{1}{N})^3|D^4z_3|_0\}^2$$

$$+ (\sqrt{q_3^*}\ \kappa_{1,0}(\tfrac{1}{N})^3|D^4z_4|_0)^2 + \{|q_2^*|\kappa_{0,1}(\tfrac{1}{N})^3|D^4z_1|_0$$

$$+ q_3^*\ \kappa_{0,2}(\tfrac{1}{N})^2|D^4z_3|_0$$

$$+ |q_2^*|(\tfrac{1}{N})^4\ \kappa_{0,0}|D^4z_3|_0\}^2, \text{ by Lemmas 1.15 and 1.16.}$$

Now since $q \in Q$ and Q is compact by hypothesis, and since z_1, z_2, z_3, z_4 all are in H^4 by the choice of $\mathscr{D}$, we have

$$|\mathscr{A}(q^*)(P^N(q^N)z-z)|^2 = O(1/N^2) \quad \text{for} \quad z \in \mathscr{D},$$

guaranteeing convergence of the second term in (3.5).

For the last term in (3.5), we have the convergence of P^N to I strongly on Z. Convergence of the first term follows explicitly from the form of the operator,

$$(\mathscr{A}(q^N) - \mathscr{A}(q^*))P^N(q^N)z = \begin{pmatrix} 0 \\ (q_1^N - q_1^*)(D^2z_1^N - Dz_3^N) \\ 0 \\ (q_3^N - q_3^*)(D^2z_3^N) + (q_2^N - q_2^*)(Dz_1^N - z_3^N) \end{pmatrix},$$

since $q^N \longrightarrow q^*$, $D^2z_1^N \longrightarrow D^2z_1$, $Dz_3^N \longrightarrow Dz_3$, $D^2z_3^N \longrightarrow D^2z_3$, $Dz_1^N \longrightarrow z_1$ and $z_3^N \longrightarrow z_3$.

Thus the hypotheses of the Trotter-Kato hold and we may conclude by it that

$$|T^N(t;q^N)z - T(t;q^*)z| \longrightarrow 0 \quad \text{uniformly on} \quad [0,T]$$

for each $z \in Z$. □

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Now, since $P^N \longrightarrow I$ strongly on Z (Lemma 3.4), Theorem 3.5 along with Proposition 1.4 guarantee the convergence of a subsequence of solutions $\bar{q}^N$ of the approximate identification problem to a solution $\bar{q}$ of the original identification problem (IDA).

We now briefly describe the concrete form this approximation takes on $Z^N = S_0^3(\Delta^N) \times S_0^3(\Delta^N) \times S_0^3(\Delta^N) \times S_0^3(\Delta^N)$. With $C_i^N \equiv B_{i,3}^N$ as the basis elements for $S_0^3(\Delta^N)$ defined in (1.15), we obtain as a basis for Z^N the set $\{\beta_i^N\}_{i=0}^{4N+3}$ where

$$\beta_i^N = \begin{cases} (C_i^N,0,0,0), & i = 0,\dots,N \\ (0,C_{i-(N+1)}^N,0,0), & i = N+1,\dots,2N+1 \\ (0,0,C_{i-(2N+2)}^N,0), & i = 2N+2,\dots,3N+2 \\ (0,0,0,C_{i-(3N+3)}^N), & i = 3N+3,\dots,4N+3. \end{cases}$$

The usual Ritz-Galerkin formulation leads to a $4N+4$-dimensional matrix system for the coefficients $w_i^N(t)$ in the expansion for $z^N(t)$ relative to the basis for Z^N. In particular, one finds (see the end of Section 2.2) that

$$z^N(t) = \sum_{i=0}^{4N+3} w_i^N(t)\beta_i^N$$

where $w^N(t) = (w_0^N(t),\dots,w_{4N+3}^N(t))$ satisfies

$$Q^N\dot{w}^N(t) = K^Nw^N(t) + R^NF$$

with $(Q^N)_{ij} = \langle\beta_i,\beta_j\rangle$, $(K^N)_{ij} = \langle\beta_i,\mathscr{A}\beta_j\rangle$, $(R^NF)_i = \langle\beta_i,F\rangle$. This becomes

$$\dot{w}^N(t) = G^Nw^N(t) + F^N \tag{3.6}$$

with

$$G^N = \begin{bmatrix} 0 & I & 0 & 0 \\ a(A_2^N)^{-1}A_1^N & 0 & -a(A_2^N)^{-1}A_3^N & 0 \\ 0 & 0 & 0 & I \\ b(A_2^N)^{-1}A_3^N & 0 & c(A_2^N)^{-1}A_1^N - bI & 0 \end{bmatrix}$$

with $(A_2^N)_{i,j} = \langle C_i^N, C_j^N\rangle_0$, $(A_1^N)_{i,j} = -\langle DC_i^N, DC_j^N\rangle_0$, $(A_3^N)_{ij} = \langle C_i^N, DC_j^N\rangle_0$,

$$(F^N)_i = \begin{cases} f_i^N, & i = N+1,\dots,2N+1 \\ 0\,, & \text{otherwise} \end{cases}$$

where $f^N = (f_0^N,\dots,f_N^N)$ corresponds to the "load" f in (3.1) and is given by $f^N = (A_2^N)^{-1}(\hat{R}^N f)$ with $(\hat{R}^N f)_i = \langle C_i^N, f\rangle_0$.

Also, since $z_1(t) \equiv y(t,\cdot)$, the approximation to the displacement y is given by

$$y^N(t,\cdot) = z_1^N(t) = \sum_{i=0}^{N} w_i^N(t)C_i^N. \tag{3.7}$$

As we did with the Euler-Bernoulli equation, we may rewrite (3.1), uusing a simple change of variables, as an abstract equation in a space which permits the use of lower order splines for z^N. Again, this leads to a different approximation.

Consider the following change of variables:

$$\begin{aligned} v_1 &= \sqrt{a}\, Dy - \sqrt{a}\, \psi \\ v_2 &= y_t \\ v_3 &= \sqrt{c}\, D\psi \\ v_4 &= \psi_t. \end{aligned}$$

Then, (3.1) becomes

$$(3.7) \qquad \frac{d}{dt}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{a} & 0 & 0 \\ \sqrt{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{c} \\ 0 & 0 & \sqrt{c} & 0 \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 & -\sqrt{a} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b/\sqrt{a} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

$$= C_1 Dv + C_2 v$$

where $v = (v_1, v_2, v_3, v_4)$. This is the standard form for a hyperbolic first-order system of partial differential equations [37, p. 108], since C_1 is symmetric.

This may be put into the previous formulation

$$\dot{z}(t) = \mathscr{A}(q)z + F(q,t), \quad t > 0$$

$$z(0) = z_0$$

with $z(t)$ now taken to be $z(t) = (z_1(t), \ldots, z_4(t)) = (v_1(t), \ldots, v_4(t))$ in $H^0 \times H^0 \times H^0 \times H^0$. In the case of boundary conditions corresponding to a fixed beam $(y(t,0) = y(t,1) = \psi(t,0) = \psi(t,1) = 0)$, we obtain $v_2(0) = v_2(1) = 0$ and $v_4(0) = v_4(1) = 0$. This formulation in a product of L^2 spaces was used to generate the "data" for numerical experiments using the method of lines package MOL1D [22].

Since this formulation does not yield $y(t,\cdot)$ directly, an auxiliary equation $\dot{z}_5 = z_2$ is included and integrated along with the system above, yielding $y(t,\cdot) = z_5(t)$.

Section 3. Numerical Results

We summarize in this section results of some of our numerical experiments using the approximate scheme presented in the previous section, based upon cubic splines. A description of the package which implements this method is the subject of Chapter 4.

As in the Euler-Bernoulli case, we took as data the values of the solution of a model of the form (3.1) whose parameters were known and sought a solution $\bar{q}^N$ of the approximate identification problem for different values of N. The "data" $\{\hat{y}_{ij}\}$, $i = 1,\ldots,r$; $j = 1,\ldots,\ell$ were generated in this case using the L^2 formulation of the Timoshenko equations (3.7) and using a general purpose computer code (MOL1D) to solve this system of first-order hyperbolic equations to obtain displacements $y(t,x)$. Then $\{\hat{y}_{ij}\}$, $i = 1,\ldots,r$; $j = 1,\ldots,\ell$ with $\hat{y}_{ij} = y(t_i,x_j)$ were used as the observations or input to the approximate identification package. For the experiments listed below, we used $\ell = 9$, $r = 10$ and generated the "observations" at $x_j = j/10$, $j = 1,\ldots,9$ for times $t_i = i/10$, $i = 1,\ldots,10$.

The optimization algorithm requires an initial guess for the parameters $q^{N,0}$, which is referred to as the "start up" values in the tables below. The values of the parameters to which the optimization algorithm converged for a given N are denoted by $\bar{q}^N$, and $J(\bar{q}^N)$ is the cost functional for those values of the parameters.

Example 3.1. We consider the motion of a beam initially at rest with fixed ends and described by the system

$$y_{tt} = q_1 y_{xx} - q_1 \psi_x + 10e^{-2t} \sin 2t$$

$$\psi_{tt} = q_3 \psi_{xx} + q_2 (y_x - \psi)$$

$$y(t,0) = y(t,1) = \psi(t,0) = \psi(t,1) = 0$$

$$y(0,x) = y_t(0,x) = \psi(0,x) = \psi_t(0,x) = 0.$$

The numerical results are given in Table 3.1.

Table 3.1

N	$\bar{q}_1^N$	$\bar{q}_2^N$	$\bar{q}_3^N$	$J(\bar{q}^N)$
6	.9882	726.19	3.6864	$.526 \times 10^{-4}$
8	.9969	781.00	3.9218	$.735 \times 10^{-5}$
10	1.0009	794.29	3.9684	$.108 \times 10^{-5}$
12	1.00036	794.90	3.9732	$.165 \times 10^{-6}$
16	1.00033	797.85	3.9883	$.256 \times 10^{-7}$
TRUE VALUE	1.0	800.	4.0	
START UP	.9	1000.	3.9	

When only one parameter was treated as unknown and the other two were held fixed at their "true values" in Table 3.1, the following results were obtained.

Table 3.2

N	$\bar{q}_1^N$	$J(\bar{q}^N)$
3	.93922	$.122 \times 10^{-1}$
4	.98452	$.105 \times 10^{-2}$
5	.99533	$.211 \times 10^{-3}$
6	.99824	$.581 \times 10^{-4}$
8	.99975	$.770 \times 10^{-5}$
TRUE VALUE	1.0	
START UP	.9	

Table 3.3

N	$\bar{q}_2^N$	$J(\bar{q}^N)$
4	814.85	$.109 \times 10^{-2}$
5	804.07	$.215 \times 10^{-3}$
6	801.47	$.587 \times 10^{-4}$
8	800.11	$.772 \times 10^{-5}$
TRUE VALUE	800.	
START UP	1200.	

Table 3.4

N	$\bar{q}_3^N$	$J(\bar{q}^N)$
3	3.77091	$.130 \times 10^{-1}$
4	3.91936	$.108 \times 10^{-2}$
5	3.97700	$.214 \times 10^{-3}$
6	3.99157	$.586 \times 10^{-4}$
8	3.99917	$.772 \times 10^{-5}$
TRUE VALUE	4.0	
START UP	3.8	

Example 3.2. We consider a clamped beam deformed to the shape $\phi(x) = \cos \lambda x + \cosh \lambda x - K(\sin \lambda x + \sinh \lambda x)$ with $\lambda \simeq 4.730$, $K = (\sin \lambda + \sinh \lambda)/(\cos \lambda + \cosh \lambda)$, then allowed to vibrate freely, which can be described by the system

$$y_{tt} = q_1 y_{xx} - q_1 \psi_x$$

$$\psi_{tt} = q_3 \psi_{xx} + q_2(y_x - \psi)$$

$$y(0,x) = \phi(x), \quad y_t(0,x) = 0, \quad \psi(0,x) = \phi'(x), \quad \psi_t(0,x) = 0,$$

$$y(t,0) = y(t,1) = \psi(t,0) = \psi(t,1) = 0.$$

The numerical results are given in Table 3.5.

Table 3.5

N	$\bar{q}_1^N$	$\bar{q}_2^N$	$\bar{q}_3^N$	$J(\bar{q}^N)$
4	1.1812	1325.3	3.6437	$.613 \times 10^{-2}$
8	.9480	1125.1	3.993	$.167 \times 10^{-2}$
10	.9938	1222.3	4.1074	$.477 \times 10^{-3}$
12	.9510	1134.7	4.1053	$.468 \times 10^{-3}$
16	.9908	1152.3	3.8875	$.242 \times 10^{-3}$
20	.9959	1181.7	3.9593	$.145 \times 10^{-3}$
24	1.0009	1193.7	3.9771	$.122 \times 10^{-3}$
TRUE VALUES	1.0	1200.	4.0	
START UP	.9	800.	3.8	

CHAPTER 4. IMPLEMENTATION OF THE APPROXIMATE IDENTIFICATION PROBLEM

Section 1. General Description of the Codes

The final chapter describes how the previously discussed methods were implemented into computer codes. While we have a different computer program for each of the methods discussed in Chapters 2 and 3, the basic structure and much of the code is the same for each program. In fact, the computer codes used for the identification problems discussed here were developed from codes written for the identification problem associated with a wave equation model examined in [7].

The computer codes were written with flexibility as the primary guiding principle. When more than one approach could be used, the one that would most easily be adapted to handle extensions or modifications of the current problem was chosen.

We describe below the algorithms in a general setting. Most of what follows is also applicable to codes used for identification problems associated with the one-dimensional hyperbolic and parabolic equations in [7] and with convection-diffusion equations discussed in [5]. All of the codes were written in FORTRAN and implemented on the IBM 370 at Brown University.

While this chapter is intended as a description of algorithms used and not as documentation for the computer codes, reference is made to the specific subroutines which implement the various algorithms.

We begin by outlining the general program for estimating the solution to the identification problem (IDA). We have seen two formulations of the Euler-Bernoulli equation and one of the Timoshenko equations for transverse vibrations of a beam. All gave rise to an abstract equation of the form

$$(4.1)\qquad \begin{aligned} \dot{z}(t) &= \mathscr{A}(q)z(t) + F(q,t), \quad t > 0 \\ z(0) &= z_0(q), \end{aligned}$$

where $z(t) = (z_1(t),\dots,z_s(t)) \in Z$, $z_0(q) = \Psi(x;q)$. In every case, we have $y(t,\cdot;q) \equiv z_1(t;q)$ where $y(t,x;q)$ is the transverse displacement of the beam at $x \in [0,1]$ and at time t associated with the parameters $q \in Q \subset R^P$.

In order to discuss the common computational features of each of the approximate identification problems discussed in Chapters 2 and 3, we note that in each case we took $Z^N = Z_1^N \times \dots \times Z_3^N$ where Z_i^N possessed a basis $\{B_i^N\}_{i=0}^N$ of appropriately modified B-splines; the basis for Z^N then was $\{\beta_i^N\}_{i=0}^{\sigma}$, $\sigma \equiv s(N+1) - 1$, where $\beta^N_{(i-1)(N+1)+j}(x) = B_j^N(x)e_i$, $i = 1,\dots,s$; $j = 0,\dots,N$, with e_i the usual basis for R^3.

In each case, the choice of $\mathscr{A}^N = P^N\mathscr{A}P^N$ led to an approximation of $y(t,x)$ of the form

$$(4.2)\qquad y^N(t,x) = \sum_{i=0}^{N} w_i^N(t)B_i^N(x)$$

where $w^N(t) = (w_1^N(t),\dots,w_\sigma^N(t))$ satisfies

$$(4.3)\qquad \begin{aligned} Q^N\dot{w}^N(t) &= K^N w(t) + R^N F(q,t), \quad t > 0, \\ w^N(0) &= P^N\Psi, \end{aligned}$$

with

$$(Q^N)_{ij} = \langle \beta_i^N, \beta_j^N \rangle, \quad (K^N)_{ij} = \langle \beta_i^N, \mathcal{A}\beta_j^N \rangle,$$
$$(R^N F)_i = \langle \beta_i, F \rangle, \quad P^N\Psi = (Q^N)^{-1} R^N(\Psi).$$

This formulation includes the following three problems from the previous chapters:

1. The Euler-Bernoulli equation with boundary conditions of type k, with approximations based upon quintic splines. In this case, $s = 2$, and $B_i^N \equiv B_{i,5}^N$ satisfying boundary conditions of type k. The inner products are those in $Z = H_k^2(\alpha) \times H^0$, namely $\langle u,v \rangle = \langle u_1, v_1 \rangle_{2,\alpha} + \langle u_2, v_2 \rangle_0$.

2. The Euler-Bernoulli equation with boundary conditions of type 1 (simply supported), with approximations based upon cubic splines. In this case, $s = 3$, and $B_i^N \equiv B_{i,3}^N$ with $B_i^N(0) = B_i^N(1) = 0$. The inner products are those in $Z = H^0 \times H^0 \times H^0$ namely $\langle u,v \rangle = \langle u_1, v_1 \rangle_0 + \langle u_2, v_2 \rangle_0 + \langle u_3, v_3 \rangle_0$.

3. The Timoshenko equations with fixed end conditions, with approximations based upon cubic splines. In this case, $s = 4$ and $B_i^N \equiv B_{i,3}^N$ with $B_i^N(0) = B_i^N(1) = 0$. The inner products are those in $H_0^1(a) \times H^0 \times H_0^1(c) \times H^0$, namely $\langle u,v \rangle = \langle u_1, v_1 \rangle_{1,a} + \langle u_2, v_2 \rangle_0 + \langle u_3, v_3 \rangle_{1,c} + \langle u_4, v_4 \rangle_0$.

Then, given data $\{\hat{y}_{ij}\}$, $i = 1,\ldots,r$; $j = 1,\ldots,\ell$ corresponding to displacements of a beam cross section located at $x_j \in [0,1]$ at time t_i, the approximate identification problem is to find $\bar{q}^N$ which minimizes

$$\begin{aligned}(4.4)\qquad \Phi^N(q) &\equiv J(q,y^N(t,\cdot;q),\hat{\eta}) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{\ell} |y^N(t_i,x_j;q) - \hat{y}_{ij}|^2,\end{aligned}$$

subject to $t \longmapsto y^N(t,\cdot)$ satisfying (4.2) and (4.3), where $\hat{\eta}_i = (\hat{y}(t_i,x_1),\ldots,\hat{y}(t_i,x_\ell))$.

So any implementation of the approximate identification problem requires an iterative procedure to minimize $\Phi^N(q)$ and a method for approximating solutions to the system of ordinary differential equations (4.3).

We discuss the common features of the computer program for each of the above problems.

As mentioned above there are two major tasks involved in the approximate identification program. The first is an unconstrained minimization problem for the sum of squares $\Phi^N(q) = \sum_{i=1}^{n} \hat{e}_i(q)^T\hat{e}_i(q)$, where $e_{ij}(q) = y^N(t_i,x_j;q) - \hat{y}_{ij}$, $\hat{e}_i = (e_{i1},\ldots,e_{im})$. and $y^N(t_i,\cdot;q)$ is the solution of (4.2) - (4.3). This task is solved efficiently by the Levenberg-Marquardt algorithm, and IMSL's version ZXSSQ has been used for this purpose. The second major task, required each time we evaluate $\Phi^N(q)$, is to solve numerically the system of ordinary differential equations (4.3). Since a wide variety of problems have been solved, some of which were stiff (for example, the Euler-Bernoulli model with structural damping coefficient $\delta > 0$), a general-purpose variable-step, variable-order method was required which would handle both stiff and non-stiff equations efficiently. Gear's algorithm, with a switch between Adams-type and backwards-difference methods was suited to this task; IMSL's version DGEAR was used. Both of these algorithms will be described below.

In addition, integration of the system (4.3) requires an efficient method for performing each of the following subtasks:

i.) Compute the basis elements. This requires an efficient method for evaluating the modified B-splines $B_i^N(x)$ and their derivatives, where $\{B_i^N(x)\}_{i=0}^N$ are splines (linear combinations of the standard B-splines) which satisfy prescribed boundary conditions.

ii.) Compute the matrices Q^N and K^N in (4.3). This involves evaluation of inner products of the form $a_{ij} = \langle D^\mu B_i^N, D^\nu B_j^N \rangle_0$, with $\mu,\nu \in \{0,1,2\}$, and storing the resulting matrix (a_{ij}) efficiently.

iii.) Computing projections. We need to project the initial function $P^N\Psi = \sum_{i=0}^{\sigma} b_i \beta_i^N$, with $b = (b_1,\ldots,b_\sigma)$ given by $b = (Q^N)^{-1}R^N(\Psi)$, and similarly we require $P^N F(t)$ where F is the non-homogeneous term in (4.1).

iv.) Compute the "spline series" $\sum_{i=0}^{N} w_i^N B_i^N(x)$.

v.) Evaluate the right-hand side of the system (4.3), namely $f(w,t) \equiv (Q^N)^{-1}\{K^N w(t) + R^N F(q,t)\}$.

Subtasks i.)-iv.) comprise the "spline package", discussed in Section 5. The subtask v.) is discussed in Section 6, where alternative ways of performing the computations are discussed. First, we give an overview of the program structure.

Section 2. Program Structure

We give an overview of the program structure. We have noted previously that macroscropically the program may be described as an optimization algorithm for minimizing $\Phi^N(q)$, combined with an approximate method for integrating the system of ordinary differential equations (4.3) in order to evaluate $\Phi^N(q)$. Since both of these tasks are solved by iterative methods and can require considerable computation, it is useful when analyzing efficiency to divide the program into a structure on three different levels:

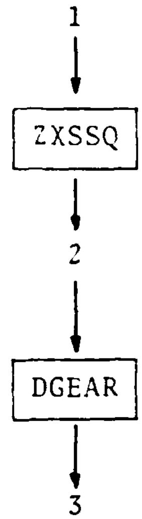
Level 1. Preprocessing, or computations which can be done prior to iteration. These include reading data $\{\hat{y}_{ij}\}$, computing and storing inner products in Q^N and K^N, and computing projections which do not depend on the parameter vector q.

Level 2. Computations inside the iterative loop for minimizing $\Phi^N(q)$. These include projections of the initial values (which may depend on q), solving the system (4.3), and computing

$$y^N(x,t) = \sum_{i=0}^{N} w_i^N(t)B_i(x)$$

Level 3. Computations which are performed at each step of the algorithm for integrating the system (4.3); these computations are those required to evaluate the right-hand side of (4.3).

The levels are connected in the following manner:



where ZXSSQ is the routine to minimize $\Phi^N(q)$, and DGEAR numerically integrates system (4.3). For a particular N (dimension of approximating subspaces), computations at level 1 need only be performed once. Computations at level 2 must be done at each iterative step of the optimization routine ZXSSQ. Since each iterative step of the optimization algorithm requires that the system (4.3) be solved (at least once per step), and the numerical algorithm for integrating (4.3) requires many evaluations of the right-hand side of (4.3), computations at level 3 are done most frequently.

In terms of routines in the packages, the routines at level 1 are those called from the main routine. These include READ, which reads input data and program parameters, and SETVL which sets the knots and computes inner products and projections which do not depend on q by calls to the spline package.

The computations at level 2 include those performed in subroutine FUNC and the routines called from FUNC, such as INIT, which computes the projections of the initial data, and CNVRT which computes $\Sigma w_i B_i^N(x)$ by calls to the spline package.

The computations at level 3 are those done exclusively in routine DEV.

Section 3. Optimization by the Levenberg-Marquardt Method

We briefly describe the Levenberg-Marquardt algorithm. We wish to find q^* which minimizes

$$\Phi(q) = \frac{1}{2} e^T(q)e(q)$$

where $e(q)$ is the vector whose components

$$e_{(i-1)\ell+j}(q) \equiv y(t_i,x_j;q) - \hat{y}_{ij},\ i = 1,\ldots,r;\ j = 1,\ldots,\ell$$

are the pointwise errors (residuals) stored as a vector. We construct a sequence of iterates $q^{(j)}$ such that $q^{(j)} \longrightarrow q^*$ by using an iteration scheme of the form

$$q^{(j+1)} = q^{(j)} + d^{(j)} \tag{4.8}$$

where $d^{(j)}$ is a solution of

$$A^{(j)}d^{(j)} = -\nabla\Phi(q^{(j)}), \tag{4.9}$$

where ∇ denotes the gradient $(\frac{\partial}{\partial q_1},\ldots,\frac{\partial}{\partial q_p})$ and $A^{(j)}$ is a matrix characteristic of the method.

If $A^{(j)} = I$, the identity matrix, then the method is steepest descent. If $A^{(j)} = \nabla^2\Phi(q^{(j)})$, where $\nabla^2\Phi$ denotes the Hessian matrix with entries $\frac{\partial^2\Phi}{\partial q_i \partial q_j}$, then the method is Newton's method obtained by truncating the Taylor series expansion after the quadratic term. While the former method is a strict descent method $(\Phi(q^{(j+1)}) \leq \Phi(q^{(j)})$ and is theoretically globally convergent, in practice convergence is both slow and unreliable [17, p. 18]. On the other hand, Newton's method converges quadratically in a neighborhood of q^* [17, p. 35], but it requires evaluation of the Hessian matrix, which makes it impractical for the problem at hand.

However, since

$$\nabla\Phi(q) = G(q)^T G(q) \tag{4.10}$$

where $G(q)$ is the Jacobian of e $(G_{ij} = \frac{\partial e_j}{\partial q_i})$, and

$$\nabla^2\Phi(q) = \Sigma e_i(q)\nabla^2 e_i(q) + G(q)^T G(q), \tag{4.11}$$

we may approximate $\nabla^2\Phi(a)$ by $G(q)^T G(q)$ for small residuals. Thus (4.8) and (4.9) with

$$A^{(j)} = G(q^{(j)})^T G(q^{(j)}) \tag{4.12}$$

is the Gauss-Newton method. Convergence of this method is at least linear in a neighborhood of q^*, and quadratic if $\Phi(q^*) = 0$ [17, p. 94]. However, this method experiences problems if G is ill-conditioned (there is no guarantee that G even be full rank away from q^*). The Levenberg-Marquardt approach is to replace (4.12) with

$$A^{(j)} = \mu^{(j)} D + G(q^{(j)})^T G(q^{(j)}), \tag{4.13}$$

where D is a positive diagonal matrix, with $\mu^{(j)} \geq 0$ chosen sufficiently large to ensure $A^{(j)}$ is positive definite and to ensure strict descent $(\Phi(q^{(j+1)}) < \Phi(q^{(j)}))$. In Marquardt and Levenberg's original scheme, $D = I$ was used. The IMSL implementation ZXSSQ uses $D_{ii} = (G^TG)_{ii}$.

Note that as $\mu^{(j)}$ becomes large, the direction of search $d^{(j)}$ approaches that of steepest descent, while $\mu^{(j)} = 0$ is the Gauss-Newton method. The method is locally convergent, and the rate of convergence is at least linear [17, p. 96].

Since $G_{ij} = \frac{\partial e_j(q)}{\partial q_i}$ is not generally available analytically, it is computed numerically. An option in ZXSSQ permits one to have the Jacobian computed numerically by

$$(4.14) \qquad \hat{G}_{ij} = \frac{e_j(q_1,\ldots,q_i+\delta_i,\ldots,q_p) - e_j(q_1,\ldots,q_i,\ldots,q_p)}{\delta_i}$$

where $\delta_i = \max(|q_i|,.1)*\sqrt{u}$, where u is the relative precision of floating point computations. While this avoids the problem of having to provide an analytic Jacobian, it costs p additional evaluations of $\Phi(q)$ to numerically compute G. Thus when $\hat{G}$ is fully evaluated, $p+1$ evaluations of $\Phi(q)$ are required per iteration.

If the Jacobian is not changing too rapidly, it is possible to approximate G using the information in the direction of the most recent step to update the Jacobian. ZXSSQ uses, when appropriate, a rank-one update of the form

$$(4.15) \qquad \hat{G}^{(j+1)} = G^{(j)} + \frac{1}{a^Ta}[e(q^{(j+1)}) - e(q^{(j)}) - G^{(j)}a]a^T$$

where $a = q^{(j+1)} - q^{(j)}$. When this approximation to the Jacobian is used, no additional evaluatons of $\Phi(q)$ are required, and so

$\Phi(q)$ will be evaluated only once in such an iteration step.

The iteration (4.8)-(4.9)-(4.13) is repeated until one of the following conditions is met:

(4.16) 1. $|\nabla\Phi(q^{(j)})| \leq \delta$

2. $|\Phi(q^{(j+1)}) - \Phi(q^{(j)})/\Phi(q^{(j)})| < \varepsilon$

3. $\Phi(q^{(j+1)})$ and $\Phi(q^{(j)})$ agree to NSIG significant digits.

Section 4. Integrating the System of Ordinary Differential Equations

We next describe very briefly the method used for integrating the system (4.3), namely IMSL's routine DGEAR [23]. For a description see [21]. We only include those details which afffect the choice of parameters for the ID package and provide some appreciation for the amount of computation required.

To abbreviate the notation, write the system (4.3) in the form:

$$\dot{w} = f(w,t), \quad w(0) = w_0. \tag{4.17}$$

The methods in DGEAR are based upon difference approximations of the form

$$w_k = \sum_{j=1}^{K_1} \alpha_j w_{k-j} + h \sum_{j=0}^{K_2} \beta_j \dot{w}_{k-j}, \quad \beta_0 > 0, \tag{4.18}$$

where α_j, β_j are constants associated with a particular method and where w_k is an approximation to $w(t_k)$, $\dot{w}_k = f(w_k,t_k)$ is an approximation to $\dot{w}(t_k)$ and h is a constant stepsize ($t_{k+1} = t_k + h$). The Adams methods of order m correspond to the values $K_1 = 1$, $K_2 = m-1$, and the backwards difference formulas (BDF) correspond to the values $K_1 = m$, $K_2 = 0$. If (4.18) is solved with all past values exact, then $w_k - w(t_k) = O(h^{m+1})$

for small h, and hence the method is said to be of order m.

As (4.18) is generally implicit, it must be solved iteratively. If we write

$$(4.19) \qquad g(w_k) = w_k - h\beta_0 f(w_k,t_k) - \sum_{1}^{K_1} \alpha_j w_{k-j} - h \sum_{1}^{K_2} \beta_j \dot{w}_{k-j},$$

then (4.18) is the implicit equation $g(w_k) = 0$. DGEAR provides a variety of methods which the user may choose to solve $g(w_k) = 0$.

First a predicted value of $w_k^{(0)}$ is obtained by an explicit method. This is equivalent to (4.18) with $\beta_0 = 0$, but DGEAR uses the Nordsicck formulation (see [21]). Then $g(w_k) = 0$ is solved iteratively by one of a class of methods of the form

$$(4.20) \qquad w_k^{(i+1)} = w_k^{(i)} - (P_k^{(i)})^{-1} g(w_k^{(i)})$$

Note if $P_k^{(i)} = (\frac{\partial g}{\partial w})\Big|_{(w_k^{(i)})} = I - h\beta_0(\frac{\partial f}{\partial w})\Big|_{(w_k^{(i)})}$, then (4.20) is Newton's method. If $P_k^{(i)} = I$, then (4.20) is functional (fixed point) iteration.

Between these two extreme lie various approximations to $(\frac{\partial g}{\partial w})\Big|_{(w_k^{(i)})}$ which are more cheaply evaluated. One choice is $P_k^{(i)} = P_k^{(0)}$; i.e., do not re-evaluate the Jacobian at each iteration. The so-called chord method uses $P_k = P_{k'}$ for some $k' \leq k$, corresponding to the parameter MITER = 1 or MITER = 2. A still less costly method, particularly where large systems are concerned, is one in which $\frac{\partial f}{\partial w}$ is approximated by a diagonal matrix whose entries are forward difference approximations to the diagonal of the Jacobian $\frac{\partial f}{\partial w}$ (MITER = 3). If $\frac{\partial f}{\partial w}$ is not available analytically, it may be computed numerically (MITER = 2).

For most of the examples considered here, the Adams methods (METH = 1) with functional iteration (MITER = 0) were most efficient. However, it should be noted that the $P^N \otimes P^N$ methods can lead to moderately stiff ODEs, as in the case where structural damping was included, in which case the BDF methods (METH = 2) and numerically computed Jacobian (MITER = 2) were used.

While the methods (4.18) are based upon constant stepsize h, DGEAR does adjust the stepsize and order. Briefly, this is accomplished by interpolating the past data $w_{k-j}, \dot{w}_{k-j}$ (using the Nordsieck array) to obtain the values of w_{k-j*} required at $t_k - jh*$ where $h*$ is the new stepsize. Following a step of size h at order m, DGEAR attempts to readjust the stepsize upward every $m+2$ steps; this is done by estimating the local truncation error at orders $m-1$, m, and $m+1$ and choosing $h*$ to be the largest permitted by these three, and the order m is reset accordingly. The estimated truncation error is effectively compared to the requested local truncation error bound TOL which is specified by the user.

One further parameter must be specified, namely an initial stepsize HØ. Since the method starts out with a first order approximation $m = 1$ (no previous history available), HØ must be small. For the ID problems considered, TOL was usually taken to be $1. \times 10^{-6}$ and HØ to be $1. \times 10^{-7}$.

Section 5. The Spline Package

The spline package contains the routines for evaluating splines and their derivatives, for computing projections onto spline subspaces, for computing inner products of the form $<D^k B^N_{i,n}, D^k B^N_{j,n}>_0$, $0 \leq k < n$, and for evaluating the "spline series" $\Sigma w_i B^N_{i,n}(x)$.

a) Evaluation of Modified B-Splines and Their Derivatives

The evaluation of the B-spline basis functions is of course fundamental to all the routines in the spline package. The B-spline basis functions were introduced in Section 3 of Chapter 1. Recall that there we defined all the B-splines in terms of a single function. We defined

$$(4.21) \qquad \bar{B}_n(x) = \delta^n (x-y)_+^{n-1} = \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x-j)_+^{n-1}.$$

Then, the B-splines of odd degree (even order) were obtained by taking $n = 2m$ ($m = 2$ for cubics, $m = 3$ for quintics) and a partition $\Delta^N = \{x_i\}_{i=1-m}^{N+m-1}$ with $x_i = i/N$; we defined

$$h = x_{i+1} - x_i$$

$$y_i = x_i - mh$$

$$\hat{B}^N_{i,n-1}(x) = \bar{B}_n((x-y_i)/h), \quad i = 1-m,\dots,N+m-1.$$

Finally, the modified B-splines $B^N_{i,n-1}(x)$ are computed by taking the appropriate linear combinations of the $\hat{B}^N_{i,n-1}$ as in (1.15)-(1.18) so that the $B^N_{i,n-1}$ satisfy the given boundary conditions.

$\bar{B}_n$ is the <u>fundamental spline</u> of order n, with knots at

$0,1,\ldots,n$, and with support on $[0,n]$. The splines $\hat{B}^N_{i,n}$ have knots Δ^N, and the support of each $\hat{B}^N_{i,2m-1}$ is $[x_{i-m},x_{i+m}]$ (or $[y_i,y_{i+2m}]$).

For the uniform partition Δ^N, two methods of evaluating the $\hat{B}^N_{i,n-1}(x)$ may be used. The first method uses the piecewise polynomial representation of $\hat{B}^N_{i,n-1}$. For example, in the case $n = 4$ (cubic splines), we obtain (see [35, pp. 89-90])

$$\hat{B}^N_{i,3}(x) = \frac{1}{h^3}\begin{cases} (x-x_{i-2})^3, \text{ if } x \in [x_{i-2},x_{i-1}] \\ h^3+3h^2(x-x_{i-1})+3h(x-x_{i-1})^2-3(x-x_{i-1})^3, \\ \qquad\qquad \text{if } x \in [x_{i-1},x_i], \\ h^3+3h^2(x_{i+1}-x)+3h(x_{i+1}-x)^2-3(x_{i+1}-x)^3, \\ \qquad\qquad \text{if } x \in [x_i,x_{i+1}], \\ (x_{i+1}-x)^3, \text{ if } x \in [x_{i+1},x_{i+2}] \\ 0, \qquad \text{otherwise.} \end{cases}$$

When each of the above expressions is evaluated using Horner's scheme, this is a very efficient method, requiring only four multiplications. The derivatives of the B-splines may be represented similarly. However, a different representation for each n and for each derivative is required.

The algorithm we have used for the computations in Chapters 2 and 3 may be used for splines of arbitrary order and for derivatives of any order. This algorithm is based upon the iterative formulas satisfied by the fundamental B-spline (see [42, p. 136])

(4.22) $$\overline{B}_n(x) = x\overline{B}_{n-1}(x) + (n-x)\overline{B}_{n-1}(x-1)$$

and

(4.23) $$D\overline{B}_n(x) = (n-1)[\overline{B}_{n-1}(x) - \overline{B}_{n-1}(x-1)].$$

The iterative formula (4.22) can be used to evaluate $B_n(x)$ for any x. Since we evaluate splines only when the values of all non-zero splines at x are required, as when computing the matrix whose entries are $\langle B^N_{i,n-1}, B^N_{j,n-1}\rangle_0$ for $0 < |i-j| \le n$, $j = 0,\ldots,N$, the algorithm may be refined to efficiently produce the values of all non-zero splines of order n at x.

First note that $\hat{B}^N_{i+j,n-1}(x) = \overline{B}_n((x-y_{i+j})/h) = \overline{B}_n((x-y_i)/h-j)$. Given n, find $i*$ such that $n-1 \le (x-y_{i*})/h \le n$, and define $\xi = (x-y_{i*})/h$. The values of the non-vanishing B-splines $\{\hat{B}^N_{i*+j,n-1}(x)\}_{j=0}^{n-1}$ then correspond to the values of $\overline{B}_n(\xi-j)$, $j = 0,\ldots,n-1$. To compute $\overline{B}_n(\xi-j)$, we note that (4.22) implies

(4.23) $$\overline{B}_j(\xi-i) = (\xi-i)\overline{B}_{j-1}(\xi-i) + (i+j-\xi)\overline{B}_{j-1}(\xi-1-i),$$
$$\text{for } j = 2,\ldots,n;\ i = n-j,\ldots,n-1.$$

Now, since $\overline{B}_1(\xi-j) = \begin{cases}1, & \text{if } j = n-1\\ 0, & \text{if } j < n-1\end{cases}$, the iterative formula (4.23) may be used to evaluate $\overline{B}_n(\xi-j)$, $j = 0,\ldots,n-1$. Then, $\hat{B}^N_{j+i*,n-1}(x) = \overline{B}_n(\xi-j)$, $j = 0,\ldots,n-1$.

If derivatives of the $\hat{B}^N_{j,n-1}$ are required, as when the matrix with entries $\langle D^2B^N_{i,5}, D^2B^N_{j,5}\rangle_0$ is to be computed, then (4.22) may be used in conjunction with the above algorithm. We compute $\overline{B}_{n-1}(\xi-j)$, $j = 0,\ldots,n-1$, then compute $D\overline{B}_n(\xi-j) = (n-1)(\overline{B}_{n-1}(\xi-j) - \overline{B}_{n-1}(\xi-j-1))$. For second derivatives, $D^2\overline{B}_n(\xi-j) = (n-1)(n-2)(\overline{B}_{n-2}(\xi-j) - 2\overline{B}_{n-2}(\xi-j-1) + \overline{B}_{n-2}(\xi-j-2))$. Note that in general to compute $D^k\overline{B}_n(\xi-j)$ $k < n-1$, we need only evaluate

$\overline{B}_{n-k}(\xi-j)$ and perform k differences on these values and multiply by $(n-1)!/(n-k-1)!$. Then, $D^k\hat{B}^N_{j+i*,n-1}(x) = D^k\overline{B}_n(\xi-j)$.

Finally, the $B^N_{j,n}$ are computed using (1.15)-(1.18) when appropriate. We note that each of these modification formulas requires only that $B^N_0, B^N_1, B^N_{N-1}, B^N_N$ be modified. Moreover, these are changed only when $x \in [0,x_2]$ or $x \in [x_{N-2},x_N]$, since the functions used to modify vanish outside these intervals. So we first check if $i = 0,1,N,N-1$ and x lies in one of the two intervals. If so, we compute

$$B^N_i(x) = \hat{B}^N_i(x) - c_{i,1}\hat{B}^N_{-1}(x) - c_{i,2}\hat{B}^N_{-2}(x), \quad i = 0,1$$

or

$$B^N_i(x) = \hat{B}^N_i(x) - c_{i,3}\hat{B}^N_{N+1}(x) - c_{i,4}\hat{B}^N_{N+2}(x), \quad i = N-1 \text{ or } N,$$

where the c_{ij} are the coefficients appearing in (1.15)-(1.18).

Remark. The algorithm presented for computing the unmodified B-splines presented above is essentially the one in [42, p. 205] and is attributed to DeBoor.

The algorithm to compute the n non-vanishing B-splines at x requires $n^2 + 2n-1$ multiplications, as compared to n^2-n multiplications for the piecewise polynomial representation. However, the flexibility in computing derivatives and splines of all order in one algorithm make the iterative algorithm attractive. Furthermore, the spline computations occur primarily at level one, and so the slight difference in efficiency has a negligible effect on overall execution time.

This algorithm appears in the various packages as SPQV, SPMVAL, or SPPVAL.

b) Computation of Inner Products

The computation of the inner products is carried out via a composite Gauss-Legendre two-point rule:

$$\int_0^1 f(x)dx = \frac{h}{2}\sum_{\nu=1}^{K} f((\nu - \tfrac{1}{2} - \sqrt{3}/6)h) + f((\nu - \tfrac{1}{2} + \sqrt{3}/6)h) \tag{4.24}$$

where $h = 1/K$, and K is chosen to be a fixed multiple of N (usually $K = 8N$). The inner products $\langle B_i^N, B_j^N\rangle_0$ are calculated by computing $x_{1\nu} = (\nu - \frac{1}{2} - \sqrt{3}/2)h$, $x_{2\nu} = (\nu - \frac{1}{2} + \sqrt{3}/2)h$, evaluating all non-zero splines at $x_{1\nu}$, $x_{2\nu}$, forming the non-zero products $B_i^N(x_{1\nu})B_j^N(x_{1\nu})$ and accumulating the products in a banded matrix (storing only nonzero subdiagonals as columns). This algorithm appears in the various packages as SETQ, SETMAT, SETMBT.

We note that for large values of N, it may be worthwhile to use the property that $\langle B_i^N, B_j^N\rangle = \langle B_{i+1}^N, B_{j+1}^N\rangle$, where $i = 2,\ldots,N-2$ for cubic splines and $i = 3,\ldots,N-3$ for quintic splines.

c) Computation of Projections

The projections in the $|\cdot|_m$-norm onto the spline subspaces are easily computed, once the inner products $\langle B_i^N, B_j^N\rangle_m$ have been computed and stored. Let P^N be a typical projection operator in the $|\cdot|_m$-norm, $m = 0,1$, or 2. Suppose we wish to compute the projection of ϕ onto $S_k^n(\Delta^N)$ where $n = 3$ $(k = 0)$ or $n = 5$ $(k = 1,2,$ or $3)$. Then the projection P^N is given by $P^N\phi = \sum_{i=0}^{N} a_i B_i^N$ where $a = (a_0,\ldots,a_N)$ is given by

$$a = (Q^N)^{-1}R^N\phi \tag{4.25}$$

with

$$(R^N\phi)_i = \langle\phi, B_i^N\rangle_m = \langle D^m\phi, D^m B_i^N\rangle_0 = \int_0^1 D^m\phi(x) D^m B_i^N(x)\,dx$$

and

$$(Q^N)_{ij} = \langle B_i^N, B_j^N\rangle_m = \langle D^m B_i^N, D^m B_j^N\rangle_0 = \int_0^1 D^m B_i^N(x) D^m B_j^N(x)\,dx.$$

The matrix Q^N is computed once via the Gauss-Legendre rule and stored; thus to compute the projections, we need only compute the inner products $\langle D^m\phi, D^m B_i^N\rangle_0$, $i = 0,\ldots,N$ via the Gauss-Legendre rule (4.24).

We note that in the case where the $|\cdot|_{m,\alpha}$-norm is used, the projections are still given by (4.25); i.e., the projections themselves do not depend on the weight α. This is because with the $|\cdot|_{m,\alpha}$-norm, $a = (\alpha Q^N)^{-1}\alpha R^N\phi = (Q^N)^{-1}R^N\phi$, where Q^N and R^N are as above. Finally, to compute the projections, we need a method for evaluating $\sum_{i=0}^{N} a_i B_i^N(x)$.

d) <u>Evaluation of the "Spline Series"</u>

We also have two methods for evaluating

$$s(x) = \sum_{i=0}^{N} c_i B_{i,n-1}^N(x). \tag{4.25}$$

First observe that because of the local support of the B-splines, that only n terms of (4.25) need be evaluated. Thus we can generate the values of the n non-vanishing B-splines at x using the previously defined algorithm and perform the linear combination. A slightly more efficient method was pointed out in [42, p. 193]. We describe this method briefly here.

Again, first find i^* such that $n-1 \leq (x-y_{i^*})/h \leq n$, and let $\xi = (x-y_{i^*})/h$. Then,

$$\hat{s}(x) = \sum_{i=0}^{N} c_i \hat{B}^N_{i,n-1}(x) = \sum_{i=0}^{n-1} c_{i^*+i} \overline{B}_n(\xi-i).$$

Define $c_i^0 = c_{i^*+i}$, $i = 0,\ldots,n$. Using (4.22), it is easily shown that

$$\sum_{i=0}^{n-1} c_i^0 \overline{B}_n(\xi-i) = \sum_{i=1}^{n-1} c_i^1 \overline{B}_{n-1}(\xi-i),$$

$$\text{where } c_i^1 = (\xi-i)c_i^0 + n-\xi+i-1)c_{i-1}^0, \quad i = .,\ldots,n-1$$

$$= \sum_{i=2}^{n-1} c_i^2 \overline{B}_{n-2}(\xi-i),$$

$$\text{where } c_i^2 = (\xi-i)c_i^1 + (n-\xi+i-2)c_{i-1}^1, \quad i = 1,\ldots,n-1$$

$$\vdots$$

$$= \sum_{i=j}^{n-1} c_i^j \overline{B}_{n-j}(\xi-j),$$

where

$$c_i^j = (\xi-i)c_i^{j-1} + (n-\xi+i-j)c_{i-1}^{j-1}. \tag{4.26}$$

For $j = n-1$, this becomes

$$= c_{n-1}^{n-1} \overline{B}_1(\xi-n+1)$$

$$= c_{n-1}^{n-1} .$$

Thus we generate the triangular array

$$\begin{array}{cccc} c_0^0 & c_1^0 & \cdots & c_{n-1}^0 \\ & c_1^1 & \cdots & c_{n-1}^1 \\ & & & \\ & & & c_{n-1}^{n-1}, \end{array}$$

using (4.26) and take $\hat{s}(x) = c_{n-1}^{n-1}$. This algorithm requires only n^2-n+2 operations. To get $s(x)$, we need to write $s(x)$ in terms of the unmodified B-splines $\hat{B}_{i,n-1}^N(x)$. We simply find $\{\hat{c}_i\}_{i=1-m}^{N+m-1}$ such that

$$\sum_{i=1-m}^{N+m-1} \hat{c}_{i+m-1}\hat{B}_{i,n-1}^N(x) = \sum_{i=0}^{N} c_i B_{i,n-1}^N(x), \tag{4.27}$$

$\hat{c}_{i+m-1} = c_i$ for $2 \leq i \leq N-2$. The others are found using (1.15)-(1.18) to write $B_{i,n-1}^N$ in terms of the $\hat{B}_{i,n-1}^N$. The computation of the coefficients $\hat{c}_i$ need only be performed when the c_i are changed. For example in the case of cubic splines modified as in (1.15), we find the requirement that

$$\sum_{i=-1}^{N+1} \tilde{c}_i\hat{B}_i^N = \sum_{i=0}^{N} c_i B_i^N$$

implies $\tilde{c}_i = c_i$ for $2 \leq i \leq N-2$, and

$$\tilde{c}_{-1}\hat{B}_{-1}^N + \tilde{c}_0\hat{B}_0^N + \tilde{c}_1\hat{B}_1^N = c_0 B_0^N + c_1 B_1^N,$$

(recall that $B_0^N = \hat{B}_0^N - 4\hat{B}_{-1}^N$ and $B_1^N = \hat{B}_1^N - \hat{B}_0^N/4$) which in turn is equal to

$$= -4c_0\hat{B}_{-1}^N + (c_0 - c_1/4)\hat{B}_0^N + c_1\hat{B}_1^N.$$

Thus we find that $\tilde{c}_{-1} = -4c_0$, $\tilde{c}_0 = c_0 - c_1/4$, $\tilde{c}_1 = c_1$, and similarly $\tilde{c}_{N-1} = c_1$, $\tilde{c}_N = c_N - c_{N-1}/4$, $\tilde{c}_{N+1} = -4c_N$.

For the quintic splines, we find that corresponding to the modified B-splines in (1.16) for simple end conditions, we obtain

$$
(4.28) \qquad
\begin{aligned}
\tilde{c}_i &= c_i, \quad 0 \le i \le N \\
\tilde{c}_{-2} &= 12c_0 - c_2 \\
\tilde{c}_{-1} &= -3c_0 - c_1 \\
\tilde{c}_{N+1} &= -3c_N - c_{N-1} \\
\tilde{c}_{N+2} &= 12c_N - c_{N-2}.
\end{aligned}
$$

Corresponding the modified B-splines in (1.19) for fixed end conditions, we obtain

$$
(4.29) \qquad
\begin{aligned}
\tilde{c}_i &= c_i, \quad 0 \le i \le N \\
\tilde{c}_{-2} &= 41.25c_0 + 32.5c_1 + 2.25c_2 \\
\tilde{c}_{-1} &= -4.125c_0 - 2.25c_1 - .125c_2 \\
\tilde{c}_{N+1} &= -4.125c_N - 2.25c_{N-1} - .125c_{N-2} \\
\tilde{c}_{N+2} &= 41.25c_N + 32.5c_{N-1} + 2.25c_{N-2}
\end{aligned}
$$

Finally, for the modified B-splines corresponding to the cantilever beam in (1.18), we obtain (4.29) except that

$$
(4.30) \qquad
\begin{aligned}
\tilde{c}_{N+1} &= 1.5c_N - .5c_{N-2} \\
\tilde{c}_{N+2} &= 3c_N - 2c_{N-1}.
\end{aligned}
$$

Section 6. Evaluation of the Right-Hand Side of the Approximating System

Since the most frequently evaluated computations occur at level 3 where we evaluate the right-hand side of the system (4.3) it is worthwhile to organize these computations efficiently.

We consider ways to evaluate the right-hand side of the system (4.3)

$$\dot{w}^N(t) = (Q^N)^{-1}\{K^N w^N(t) + R^N F(q,t)\}.$$

Consider first the homogeneous part, $(Q^N)^{-1}K^N w$. Note that Q^N will have a block diagonal structure

$$Q^N = \begin{bmatrix} Q_1 & & & & \\ & Q_2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & Q_s \end{bmatrix} .$$

To be concrete, let us restrict ourselves to the case $s = 2$, which is the case of the Euler-Bernoulli formulation. In this case,

$$Q^N = \begin{bmatrix} Q_1^N & 0 \\ 0 & Q_2^N \end{bmatrix}, \quad Q_i^N \text{ is } (N+1) \times (N+1)$$

and

$$K^N = \begin{bmatrix} 0 & K_1^N \\ K_2^N & K_3^N \end{bmatrix}, \quad K_i^N \text{ is } (n+1) \times (N+1).$$

Each of the submatrices Q_i^N and K_i^N has a banded structure; each will be 7-banded when cubic splines are used and 11-banded when quintic splines are used. Moreover, since each is a symmetric matrix, these can be stored efficiently by storing the subdiagonals and the diagonal as columns of a matrix which is $(N+1) \times 4$ (or $(N+1) \times 6$ for quintic splines).

Two ways can be used to evaluate a general term $(Q_i^N)^{-1}K_j^N v$, where $v = (v_1,\ldots,v_{N+1})$. One way is to compute $A = (Q_i)^{-1}K_j^N$ at level one and save the matrix A. Then when evaluating the right-hand side at level 3, we need only do a matrix multiplication Av. However, A will in general be a full matrix (no banded or sparse structure), and so this multiplication requires $(N+1)^2$ operations.

Another way to do this, which preserves the banded structure, is to factor $Q_i^N = LL^T$ where L is lower triangular, at level 1 by the Cholesky algorithm, and store the factor L as a banded matrix (requiring $(N+1) \times 4$ or $(N+1) \times 6$ locations). Then at level 3, we compute

i) $\zeta = K^N v$

ii) $L\phi = \zeta$

iii) $L^T\chi = \zeta$.

Step (i) requires 7N-5 operations (11N-9 for quintics), steps (ii) and (iii) each require 4N-1 (6N-20 for quintics) to backsolve a banded triangular system of algebraic equations. When done in this manner, the computation of $(Q_i^N)^{-1}K_j^N v$ requires 15N-7 operations, $N \geq 4$, when cubic splines are used, and 23N-49 operations, $N \geq 6$, when quintic splines are used. Thus for large values of N ($N \geq 16$), there is a clear advantage to the second approach.

The Cholesky decomposition of the matrix Q_i^N into LL^T was carried out by IMSL routine LUDAPB, and the backsolution (ii) and (iii) by IMSL routine LUELPB.

We can now compare the computational efficiency of the two approximations discussed for the Euler-Bernoulli beam. In order to make a comparison, consider the case when $\gamma = 0$ (no viscous damping).

For the quintic spline formulation, we had

$$Q^N = \begin{bmatrix} Q_1^N & 0 \\ 0 & Q_2^N \end{bmatrix}$$

$$K^N = \begin{bmatrix} 0 & K_1^N \\ \alpha K_2^N & \delta K_3^N \end{bmatrix},$$

with $Q_1^N = K_1^N$. So $(Q^N)^{-1}K^N$ had the structure

$$\begin{bmatrix} 0 & I \\ \alpha Q_2^{-1}K_2^N & \delta Q_2^{-1}K_3^N \end{bmatrix}.$$

So with $u = (u_1,\ldots,u_{N+1})$, $v = (v_1,\ldots,v_{N+1})$, we compute

$$(Q^N)^{-1}K^N\begin{pmatrix} u \\ v \end{pmatrix} = \begin{bmatrix} v \\ Q_2^{-1}(\alpha K_2^N u + \delta K_3^N v) \end{bmatrix}$$

requiring about $34N$ operations to evaluate when $\delta > 0$ and about $29N$ operations when $\delta = 0$.

For the cubic spline formulation, we obtained

$$(Q^N)^{-1}K^N = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & -\alpha(A_2^N)^{-1}A_1^N \\ 0 & (A_2^N)^{-1}A_1^N & 0 \quad 0 \end{bmatrix},$$

when γ,δ are both zero. The evaluation of $(Q^N)^{-1}K^N\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ requires about $30N$ operations. In addition, an extra set of $N+1$ equations has been used, compared to the quintic spline formulation. Therefore, based upon this and our computational experience there seems to be no advantage in going to the cubic spline formulation.

To evaluate the non-homogeneous term, note $(Q^N)^{-1}R^N F(q,t) = P^N F(q,t)$. The projections may be computed as in Section 5. If the load $f(t,x;q)$ can be written as $g(x)h(t;q)$, then $F(q,t)$ has the form $(0,g)^T \cdot h(t;q)$ and the projections $P^N(0,g)^T$ may be computed at level 1 (once for any given N) and stored.

CHAPTER 5. CONCLUSIONS

We have presented numerical methods for parameter identification of several constant parameters appearing in the Euler-Bernoulli and Timoshenko equations for transverse vibrations of a beam. The practical utility of our approach is supported by our computational experience on a large number of examples, a sample of which has been included here. In most cases of interest we have given a complete treatment of our methods including numerical algorithms, convergence proofs, and numerical results.

The fundamental ideas upon which our convergence results are based, involving the use of a semigroup theoretic approach for the approximation of identification problems governed by partial differential equations (distributed parameter systems), first appeared in [11]. These methods essentially involve writing the initial-boundary value problem as an abstract equation

$$(5.1) \qquad \dot{z}(t) = \mathcal{A}(q)z(t) + F(q,t)$$

in a Hilbert space Z, where $\mathcal{A}(q)$ is the generator of a C_0 semigroup in Z, and approximating the generator $\mathcal{A}(q)$ by the operator $\mathcal{A}^N(q) = P^N\mathcal{A}(q)P^N$, with P^N the projection onto a finite dimensional subspace Z^N spanned by splines. The resulting system of ordinary differential equations

$$(5.2) \qquad \dot{z}^N(t) = \mathcal{A}^N(q)z^N(t) + F^N(q,t)$$

is used to approximate solutions of (5.1).

Instead of seeking a $\bar{q}$ which minimizes a cost functional $J(q,z(\cdot,q))$ over mild solutions of (5.1), we obtain estimated values for $\bar{q}$ by seeking a $\bar{q}^N$ which minimizes the cost functional $J(q;z^N(\cdot;q))$ over solutions $z^N(\cdot;q)$ to (5.2).

The idea of estimating solutions to the identification problem is not new. The notion of approximating solutions of the partial differential equation and performing the optimization on the approximate solutions to obtain parameter estimates has been used extensively. We refer the reader to the survey articles [2], [26], [32], and [36] to see the variety of approaches which have been used. We only discuss some of the relevant material here.

We note that many investigators have used finite-difference methods, modal approximations, and the methods of lines to approximate solutions of the partial differential equation (see Table 1 in [26]) in the context of parameter identification. Galerkin methods, to which our methods are closely related, have also been used. In [33], Galerkin methods are used for the heat equation employing a basis of polynomials which satisfy the boundary conditions and which are orthogonalized via Gram-Schmidt.

While most of the work in the surveys deals with the estimation problem for the heat equation, several authors have proposed numerical methods for estimating a single parameter in the beam equation

$$y_{tt} = -\alpha y_{xxxx}. \tag{5.3}$$

In [19] an example is given where this is done using finite differences to approximate solutions to (5.3). We have found no results in the literature for estimating structural or viscous damping coefficients or the parameters in the Timoshenko equation.

Cubic splines have also been used in the context of estimation of parameters (see surveys above). In [43] they are used for the one-dimensional heat equation to obtain a "lumped system" of ordinary differential equations which approximate the partial differential equation, by collocating with cubic splines in the spatial variable. We believe our approximations using $P^N \mathscr{A}(q) P^N$ with cubic and quintic splines are new.

Little work has been done in proving convergence of parameter estimation schemes, and we have found no such theoretical work in the literature for methods comparable to ours. The literature on parameter estimation consists mainly of proposed numerical methods with test results for a simple example estimating a single identification of many parameters and have provided proofs of convergence for our methods.

While some authors have investigated the identification of coefficients which are a function of the spatial variable (see Table 1 in [26]), we have restricted our attention here to the case of constant parameters. Our methods do carry over to the case of spatially varying coefficients, and this has been done for a special case of the convection-diffusion equation (see [5, p. 22] for a discussion). We have not treated the questions of observability on identificability, but have chosen to emphasize the convergence of the parameter estimates to the solution of the identification problem.

In the case of deterministic data (noise free), most authors have used a steepest descent method to perform the optimization (see Table 1 in [26] and other surveys). It is well-known that steepest descent can exhibit oscillatory behavior and perform badly in practice [17, p. 18]. For this reason, we have chosen the more robust Levenberg-Marquardt method.

We have used a discrete least-squares fit-to-data criterion (cost functional). Many other type of cost functionals have been used also. In particular, cost functionals of the form

$$J(q,y(\cdot,\cdot;q)) = \sum_{i=1}^{\ell} \int_0^T |y(t,x_i) - \eta_i(t)|^2 dt \tag{5.4}$$

(continuous measurements in time from a finite number of sensors) have been considered. The methods we have proposed extend directly to such functionals. In fact, ours may be considered a digitized version of (5.4), where only a finite number of time samples are recorded.

We have only used state observations corresponding to displacements $y(t_i,x_j)$ to simplify the discussion. It should be clear that we could also have used data of the form $y_t(t_i,x_j)$ (velocity measurements) or data of the form $y_x(t_i,x_j)$ (from strain gauges) as well, with only slight modifications in the arguments and in the computer codes.

The theory that we have developed (following [11]) is the semidiscrete approximations (5.2) (i.e., continuous in time). This serves to decouple the analysis for the spatial approximation from the time discretization that is employed in practice. This is somewhat necessary because we have used variable-step/

variable order methods (Gear's package) to integrate the approximating system of ordinary differential equations to a specified local error tolerance. While a full analysis of such a fully discretized method would be difficult, fully discrete methods based on Padé approximations to $\exp(\mathscr{A}^N(q)t)$ have been analyzed for functional differential equations [39] and this analysis is being carried over to our methods for partial differential equations (for fixed step length Δt).

The semigroup theoretic approach (and in particular the Trotter-Kato theorem) provides a simple approach to obtaining convergence results. Other methods have been used and may be more powerful in obtaining results in a broader class of equations. For example, [8] have used a weak formulation and Gronwall-type estimates to obtain convergence results for estimation problems involving a class of non-autonomous equations.

We also note that in [16], methods to approximate solutions to an elliptic equation are proposed, and an a priori estimate of the error $|\bar{q}^N - \bar{q}|$ is derived. The Trotter-Kato approach does not appear to yield such estimates in any easy fashion.

In short, we have presented a unified treatment of a class of parameter estimation problems involving certain beam equations. Our treatment includes new methods of approximation (based on the classical approximations $P^N\mathscr{A}(q)P^N$), proofs of convergence of the approximate identification problem, and numerical results.

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